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<td><strong>Author(s)</strong></td>
<td>KOSHITANI, SHIGEO</td>
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ON GROUP ALGEBRAS OF FINITE GROUPS

Shigeo KOSHITANI

Department of Mathematics, Faculty of Science
Chiba University, Chiba-city, 260, Japan

In this note we study the group algebra KG of a finite p-solvable group G over a field K of characteristic p > 0. Let J(KG) be the Jacobson radical of KG, and let t(G) be the least positive integer t such that \( J(KG)^t = 0 \). Since \( J(EG) = E \otimes_K J(KG) \) for any extension field E of K, we may assume that K is algebraically closed. We would like to know the relation between t(G) and the structure of G. When t(G) ≤ 3, p-solvable groups G are completely determined by D.A.R. Wallace ([9], [10]) and K. Motose and Y. Ninomiya [7]. The purpose of this note is to determine the structure of p-solvable groups G with t(G) = 4 under the assumption that \( O_{p'}(G) \) are abelian.

We shall use the following notation. For a positive integer n let \( S_n \) and \( A_n \) be the symmetric group and the alternating group of degree n, respectively. Let \( O_{p'}(G) \) and \( O^{p'}(G) \) be the maximal normal subgroup of G of order prime to p and the minimal normal subgroup of G of index 1.
prime to \( p \), respectively. Following custom we write \( O(G) \) and \( O'(G) \) for \( O_2(G) \) and \( O_2'(G) \), respectively. For a ring \( R \) and a positive integer \( n \) let \((R)_n\) be the ring of all \( n \times n \) matrices with entries in \( R \). We use the other notation following Gorenstein’s book [3].

By making use of [9, Theorem], [2, Theorem 1] and [10, Theorem 3.3] we have

**Proposition 1.** If \( G \) is a finite \( p \)-solvable group with a \( p \)-Sylow subgroup \( P \) and if \( t(G) = 4 \), then \( p = 2 \) and one of the following holds;

(i) \( P \) is cyclic of order 4,

(ii) \( P \) is elementary abelian of order 8,

(iii) \( G/O(G) \cong S_4 \).

**Remark 1.** The converse of Proposition 1 does not hold in general (see Motose’s example [6, Example 2]). However, the following holds.

**Proposition 2.** If \( p = 2 \) and if \( G \) is a finite 2-solvable group with a 2-Sylow subgroup \( P \) which satisfies one of the following;

(i) \( P \) is cyclic of order 4,

(ii) \( P \) is elementary abelian of order 8,

(iii) \( G = S_4 \),

then \( t(G) = 4 \).
Because of Propositions 1 and 2 we assume in the rest of this note that
\[ p = 2 \quad \text{and} \quad G/O(G) \cong S_4. \]

Then a 2-Sylow subgroup \( P \) of \( G \) is dihedral of order 8. Thus, by [3, Theorem 7.7.3], \( P \) has subgroups \( X \) and \( Y \) such that \( X \) and \( Y \) are both noncyclic of order 4, \( X \not\cong Y \), \( |N_G(X) : C_G(X)| = 6 \) and \( |N_G(Y) : C_G(Y)| = 2 \).

By [4, V 25.12 Satz, V 25.7 Satz und V 25.3 Satz], [8, Lemma 2.1] and [11, Proposition 3.2], we have

Lemma 1. If \( U \) is a subgroup of \( S_4 \) and if \( K^cU \) is a twisted group algebra of \( U \) over \( K \) with respect to the factor set \( c \), then \( K^cU \cong KU \) as \( K \)-algebras.

By making use of Lemma 1, [5, Theorem 2] and [1] we obtain the following two lemmas.

Lemma 2. \( t(G) = 4 \) if and only if \( t(N_G(X)) = 4 \).

Lemma 3. If \( X \not\cong G \), then
\[
K^G \cong \bigoplus_{i=1}^{m} (K^G S_4^i \alpha_i) \oplus \bigoplus_{j=1}^{n/2} (K^G A_4^j \beta_j) \oplus \bigoplus_{k=1}^{u/3} (K^G F_k \gamma_k) \oplus \bigoplus_{\ell=1}^{v/6} (K^G X \delta_\ell)
\]
as \( K \)-algebras for positive integers \( \alpha_i, \beta_j, \gamma_k \) and \( \delta_\ell \) where \( m, n, u \) and \( v \) are the numbers of irreducible
complex characters $\psi$ of $O(G)$ such that $I_G(\psi)/O(G) \cong S_4$, $A_4$, $P$ and $X$, respectively, and $I_G(\psi)$ is the inertia group of $\psi$ in $G$.

From the above lemmas we have the following main result.

Theorem. Let $M = O'(N_G(X))$. If $O(M)$ is abelian, then the following are equivalent:

1. $t(G) = 4$.
2. $t(M) = 4$.
3. $|C_M(P)| = 2$ where $P$ is a 2-Sylow subgroup of $M$.
4. When $g \in M$ such that $|gO(M)| = 3$ in $M/O(M)$, we have $g \in C_M(O(M))$.

Remark 2. In Theorem for the case where $O(M)$ is nonabelian (2) and (3) are not equivalent in general.

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