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Some Topics in Integral Group Rings

Ken-Ichi Tahara, Kazunori Horibe
Masao Shimizu and Akinori Hosomi
Aichi University of Education

We shall talk about two topics in integral group rings: the first is a result on the augmentation quotients of integral group rings, and the second is one on the circle groups.

I. Augmentation Quotients of Integral Group Rings

Let G be a group, $\mathbb{Z}G$ its integral group ring of G over the ring $\mathbb{Z}$ of all rational integers, and $\Delta = \Delta(G)$ the augmentation ideal of $\mathbb{Z}G$. Consider the augmentation quotients $Q_n(G) = \Delta^n/\Delta^{n+1}$ for all natural integers n.

(1) Bachmann-Grünenfelder [1] proved the following:

Let G be a finite nilpotent group of class c. Then there exist natural numbers n_o, π such that π divides the least common multiple of integers $1, 2, \cdots, c$, and $Q_{n+\pi}(G) \cong Q_n(G)$ for all $n \geqq n_o$.

The proof of the existence of n_o and π is a consequence of Jordan-Zassenhaus' theorem:

$$|\{M : G\text{-module of } \mathbb{Z}\text{-rank } (|G|-1)\}/\cong| < \infty$$
$$\text{VII}$$
$$|\{\Delta^n \mid n \geqq 1\}/\cong|$$

Then there exist natural integers n_o, π such that $\Delta^{n_o+\pi} \cong \Delta^{n_o}$. We have the following commutative diagram of G-module homomorphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(G, \Delta^{n_o+\pi}) & \longrightarrow & \Delta^{n_o+\pi} \otimes \Delta & \longrightarrow & \Delta^{n_o+\pi+1} & \longrightarrow 0 & : \text{exact} \\
 & & \downarrow\wr & & \downarrow\wr & & \downarrow & & \\
0 & \longrightarrow & H_1(G, \Delta^{n_o}) & \longrightarrow & \Delta^{n_o} \otimes \Delta & \longrightarrow & \Delta^{n_o+1} & \longrightarrow 0 & : \text{exact}
\end{array}$$

and hence $\Delta^{n_o+\pi+1} \cong \Delta^{n_o+1}$. Thus we get $\Delta^{n+\pi} \cong \Delta^n$ and hence $Q_{n+\pi}(G) \cong Q_n(G)$ for all $n \geqq n_o$. The proof of the result that π divides the least common multiple of $1, 2, \cdots, c$ is not easy and is omitted here. The key point is that the Poincare series of G is a rational function. See [1] for details.

For any finite group G, the structure of $Q_1(G)$, $Q_2(G)$, $Q_3(G)$ and $Q_4(G)$ is completely determined ([8, 9, 10, 11]), and Losey-Losey [6] called the sequence $Q_{n_o}(G)$, $Q_{n_o+1}(G)$, $\cdots$, $Q_{n_o+\pi-1}(G)$ the stable bahavior of $Q_n(G)$.

(2) Passi [7] determined completely the stable behavior for all cyclic groups and all elementary abelian p-groups.

(3) Losey-Losey [5] determined elegantly the stable behavior for any finite p-group G with the lower central series $G =$

$G_1 \supset G_2 \supset \cdots \supset G_c \supset G_{c+1} = 1$ such that $G_i^p \subseteq G_{ip}$ for all $i \geqq 1$.

(4) Losey-Losey [6] determined also the stable behavior for all p-groups of order p^3.

(5) Problem. Determine the stable bahavior for all p-groups of order p^4.

For this problem, K. Horibe determined the stable behavior for six some p-groups of order p^4 with an odd prime p.

We explain how to determine the stable behavior for these groups using the following group G as an example:

$$G = \langle x, y, z \mid x^p = y^p = z^{p^2} = [x, z] = [y, z] = 1,\ [x, y] = z^p \rangle .$$

Any element g of G can be uniquely written as follows

$$g = x^h y^i z^j (z^p)^k, \quad 0 \leqq h, i, j, k \leqq p.$$

Here x, y and z have the weight 1, and z^p has the weight 2 with respect to the lower central ceries. In general we consider 4-sequences $\alpha = (h, i, j, k)$ with non-negative integers h, i, j, k and the proper products

$$P(\alpha) = (x-1)^h (y-1)^i (z-1)^j (z^p-1)^k.$$

Now α is called basic if all integers h, i, j, k are smaller than p. We partition the set of all 4-sequences into the following four classes:

$\Psi(\mathrm{I})=\{(0, 0, j, 0) \mid j \geqq 1\}$

$\Psi(\mathrm{II})=\{(0, 0, j, k) \mid k \geqq 1\}$

$\Psi(\mathrm{III})=\{(h, i, j, 0) \mid (h, i)\neq(0, 0)\}$

$\Psi(\mathrm{IV})=\{(h, i, j, k) \mid (h, i)\neq(0, 0), k \geqq 1\}.$

For simplicity we put $\Psi = \Psi(\mathrm{II})\cup\Psi(\mathrm{III})\cup\Psi(\mathrm{IV})$. then we have the following:

$$\begin{cases} \alpha \in \Psi(\mathrm{II}) \Longrightarrow & p^m P(\alpha) \in \Delta^{j+(k-1)p+2+mp(p-1)} \\ \alpha \in \Psi(\mathrm{III}) \Longrightarrow & p^m P(\alpha) \in \Delta^{h+i+j+m(p-1)} \\ \alpha \in \Psi(\mathrm{IV}) \Longrightarrow & p^m P(\alpha) \in \Delta^{h+i+j+kp+mp(p-1)} \end{cases}.$$

We write $N(\alpha, m)$ the index of Δ in the above each case. For any $n \geqq 1$ we put

$$m_\alpha(n) = \min\{ m \in \mathbb{Z} \mid m \geqq 0,\ N(\alpha, m) \geqq n\}.$$

Then we can easily get

$$m_\alpha(n) = \begin{cases} \left[\dfrac{n-j-(k-1)p-2-1}{p(p-1)} + 1\right] \\ \left[\dfrac{n-h-i-j-1}{p-1} + 1\right] \\ \left[\dfrac{n-h-i-j-kp-1}{p(p-1)} + 1\right] \end{cases}$$

where $[u]=\begin{cases} 0 & (u<0) \\ \text{the largest integer} \leqq u & (u \geqq 0) \end{cases}.$

Now we put $T_\alpha(n)$ the rational number which is appeared in the bracket [ ] of the above formula on $m_\alpha(n)$. The following is the key proposition.

Proposition. For any $n \geqq 1$,

$$\{(z-1)^n, (z-1)^{n+1}, \cdots, (z-1)^{n+p-2}, p^{m_\alpha(n)} P(\alpha), \alpha \in \Psi\text{:basic}\}$$

is a $\mathbb{Z}$-free basis of Δ^n.

Corollary. For any $n \geqq 1$,

$$\{p(z-1)^n, (z-1)^{n+1}, \cdots, (z-1)^{n+p-2}, p^{m_\alpha(n)} P(\alpha), \alpha \in \Psi\text{:basic}\}$$

is a $\mathbb{Z}$-free basis of Δ^{n+1}.

Therefore $Q_n(G) = \Delta^n/\Delta^{n+1}$ is an elementary abelian p-group of rank s_n,

$$s_n = 1 + \#\{\text{basic } \alpha \in \Psi \mid m_\alpha(n+1) > m_\alpha(n)\}.$$

On the other hand we put

$$n_\circ = \min\{n \mid T_\alpha(n) \geqq 0, \ \alpha \in \Psi\text{: basic}\}$$

then we have easily $n_\circ = 3p-2$. By calculating the number of basic α such that $m_\alpha(n+1) > m_\alpha(n)$ in each case, we have for any $n \geqq n_\circ$,

$$\begin{aligned} s_n &= 1 + 1 + (p^2+p) + (p^2-1) \\ &= 2p^2 + p + 1, \end{aligned}$$

and hence $\pi = 1$. Thus we have

<u>Theorem</u> If $G = \langle x, y, z \mid x^p = y^p = z^{p^2} = [x, z] = [y, z] = 1,$ $[x, y] = z^p \rangle$, then $n_o = 3p-2$, $\pi = 1$ and

$$Q_n(G) = C_p^{2p^2+p+1}, \quad n \geq n_o,$$

where C_p is the cyclic group of order p.

We summarize all the stable behavior for six groups of order p^4 which we could compute completely.

(6) These six groups have the following stable behavior :

a) $G = C_p \times \langle x, y, z \mid x^p = y^p = z^p = [x, z] = [y, z] = 1, \ [x, y] = z \rangle$

$n_o = 3p-2, \quad \pi = 2$

$Q_{3p-2}(G) = C_p^{\frac{1}{2}(p+1)(p^2+p+1)}, \quad Q_{3p-1}(G) = C_p^{\frac{1}{2}(p+1)(p^2+p+1)+1}$

b) $G = \langle x, y, z \mid x^p = y^p = z^{p^2} = [x, z] = [y, z] = 1, \ [x, y] = z^p \rangle$

$n_o = 3p-2, \quad \pi = 1$

$Q_{3p-2}(G) = C_p^{2p^2+p+1}$

c) $G = C_p \times \langle x, y \mid x^p = y^{p^2} = 1, \ [x, y] = y^p \rangle$

$n_o = 3p-2, \quad \pi = 1$

$Q_{3p-2}(G) = C_p^{2p^2+p+1}$

d) $G = C_p \times C_p \times C_{p^2}$

e) $G = \langle x, y, z \mid x^p = y^p = z^{p^2} = [x, y] = [x, z^p] = 1,\ [x, z] = y,\ [y, z] = z^p \rangle$

$n_\circ = 4p-3, \quad \pi = 2$

$Q_{4p-3}(G) = C_p^{\frac{1}{2}(3p^2+2p+1)}, \quad Q_{4p-2}(G) = C_p^{\frac{1}{2}(3p^2+2p+1)+1}$

f) $G = \langle x, y, z \mid x^p = y^p = z^{p^2} = [x, y] = [y, z] = 1,\ [x, z] = y \rangle$

$n_\circ = 4p-3, \quad \pi = 2$

$Q_{4p-3}(G) = C_{p^2} \times C_p^{\frac{1}{2}(3p^2+p-1)-1}, \quad Q_{4p-2}(G) = C_{p^2} \times C_p^{\frac{1}{2}(3p^2+p-1)}$.

II. Circle Groups

We shall define a circle group. Let R be a ring. For any elements a, b of R we put

$$a \circ b = a + b + ab .$$

If the set R is a group under the operation $\circ$, we call $(R, \circ)$ the circle group of the ring R. We have easily the following.

If $R \supset R^2 \supset \cdots \supset R^n \supset R^{n+1} = 0$ is a nilpotent ring of index $n+1$, then $(R, \circ)$ is a nilpotent group of class $\leq n$.

Problem. Which groups is realized as circle groups of rings ? Moreover when groups are restricted to nilpotent groups, which nilpotent groups of class n are realized as circle groups of nilpotent rings of index $(n+1)$?

(7) The case that groups are abelian.

Let G be an abelian group. Then G can be made a ring by defining $gh=0$ for all g, h of G, and the group G is isomorphic to the circle group $(G, \circ)$.

(8) The case that groups are nilpotent of class 2.

Hales-Passi [2] proved the following.

Proposition. The following are equivalent:

(i) G is the circle group of a nilpotent ring of index 3

(ii) there exists a normal subgroup N such that

a) $Z(G) \geqq N \geqq D_2(G) = G_2$, the second dimension subgroup

b) the short exact sequence

$$0 \longrightarrow N \xrightarrow{i} \frac{\Delta^2(G)+\Delta(N)\mathbb{Z}G}{\Delta^3(G)+\Delta(N)\Delta(G)} \xrightarrow{j} Q_2(G/N) \longrightarrow 0$$

splits, where $i(x)=\overline{x-1}$, $j(\overline{(g_1-1)(g_2-1)}) = (\bar{g}_1-\bar{1})(\bar{g}_2-\bar{1}) + \Delta^3(G/N)$.

We want to extend the result for nilpotent groups of higher class. In fact we could do for the case of class 3. This is the master thesis of A. Hosomi.

Before we mention the result for the case of class 3, we introduce the notation we use here. Let G be a group with a normal subgroup N, and ϕ_N be a canonical homomorphism from G to G/N. We can extend ϕ_N a ring homomorphism $\tilde{\phi}_N : \mathbb{Z}G \longrightarrow \mathbb{Z}(G/N)$ by linearity.

Theorem. The following are equivalent:

(i) G is the circle group of a nilpotent ring of index 4

(ii) there are two normal subgroups M, N with $M \geqq N$ which satisfy a) $\sim$ e):

a) $Z(G) \geqq N \geqq D_3(G)$, the third dimension subgroup

b) $Z(G/N) \geqq M/N \geqq D_2(G/N)$, the second dimension subgroup

c) the short exact sequence

$$0 \longrightarrow N \underset{\psi_1}{\overset{i_1}{\rightleftarrows}} \frac{\Delta^3(G)+\Delta(N)\mathbb{Z}G}{\Delta^4(G)+\Delta(N)\Delta(G)} \xrightarrow{j_1} Q_3(G/N) \longrightarrow 0$$

splits, where $i_1(x) = \overline{x-1}$, $j_1(\overline{(g_1-1)(g_2-1)(g_3-1)}) = (\bar{g}_1-\bar{1})(\bar{g}_2-\bar{1})(\bar{g}_3-\bar{1}) + \Delta^4(G/N)$, and ψ_1 is the splitting homomorphism of i_1

d) the short exact sequence

$$0 \longrightarrow M/N \underset{\psi_2}{\overset{i_2}{\rightleftarrows}} \frac{\Delta^2(G/N)+\Delta(M/N)\mathbb{Z}(G/N)}{\Delta^3(G/N)+\Delta(M/N)\Delta(G/N)} \xrightarrow{j_2} Q_2(G/M) \longrightarrow 0$$

splits, where $i_2(x) = \overline{x-1}$, $j_2((\bar{g}_1-\bar{1})(\bar{g}_2-\bar{1})) = (\bar{\bar{g}}_1-\bar{\bar{1}})(\bar{\bar{g}}_2-\bar{\bar{1}}) + \Delta^3(G/M)$, and ψ_2 is the splitting homomorphism of i_2.

e) there is an ideal L_2 of $\mathbb{Z}G$ contained in $\Delta^2(G) + \Delta(M)\mathbb{Z}G$ such that $\operatorname{Ker} \psi_2 = \widetilde{\phi}_N(L_2)/\Delta^3(G/N)+\Delta(M/N)\Delta(G/N)$, and there is a homomorphism $\psi_3 : L_2 + \Delta(N)\mathbb{Z}G \longrightarrow N$ with $\psi_3(\alpha) = \psi_1(\alpha + \Delta^4(G) + \Delta(N)\Delta(G))$ for all $\alpha \in \Delta^3(G) + \Delta(N)\mathbb{Z}G$ such that $\operatorname{Ker} \psi_3$ is an ideal of $\mathbb{Z}G$.

Roughly speaking, a group G is the circle group of a nilpotent ring of index 4 if and only if the group G/N

satisfies the criterion of Hales-Passi and the short exact sequence of degree 4 for G itself splits.

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