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On splitting modules for group extensions

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Introduction.

Let $H^n(G,A)$ be the n-th cohomology group of a group G in a G-module A. A G-module B containing A is said to be a splitting module for $\alpha \in H^n(G,A)$, if for the inclusion map $i; A \longrightarrow B$ we have $i^*(\alpha) = 0$, where i^* ; $H^n(G,A) \longrightarrow H^n(G,B)$ is the homomorphism of cohomology groups induced from i.

It is known that for each n and $\alpha \in H^n(G,A)$ there exists a splitting module. For an application, we are particularly interested in a splitting module for $\alpha \in H^2(G,A)$. A wellknown construction of such a module is as follows (see, E.Weiss: Cohomology of Groups, Acad.Press, 1969).

Let $a_{x,y}$ be a normalized 2-cocycle which represents α. As the additive group of B we take the direct sum of A and a free abelian group whose basis consists of symbols b_x, one for each element x of G except the identity. For notational convenience we define b_1 to be the zero element of B. We extend the operation of G from A to B in just such a way that the 2-cocycle $a_{x,y}$ becomes the coboundary of the 1-cochain b_x. This is done by defining

$$xb_y = b_{xy} - b_x + a_{x,y} \quad \text{for } x,y \in G.$$

In this note, we shall show that a splitting module for $\alpha \in H^2(G,A)$ can be also constructed from the group ring Z[U] of a group extension U corresponding to α. Moreover if G is a finite group, we shall prove several propositions on properties of this splitting module. Our proofs of these propositions seem to be more simpler than the proofs using the splitting module formally constructed in above.

1. An alternative construction of a splitting module.

Let U be a group. We denote by Z[U] the integral group ring of U and by I(U) the augmentation ideal of U. Given a cohomology class $\alpha \in H^2(G,A)$, we take a group extension

$$1 \longrightarrow A \longrightarrow U \longrightarrow G \longrightarrow 1$$

corresponding to α , where the G-module A is viewed as a multiplicative subgroup of U. Consider the exact sequence

$$0 \longrightarrow I(U) \longrightarrow Z[U] \longrightarrow Z \longrightarrow 0$$

of A-modules. Taking homology, we get an exact sequence

$$0 \longrightarrow H_2(A,Z) \xrightarrow{i} H_1(A,I(U)) \xrightarrow{j} H_1(A,Z[U]),$$

where $H_2(A,Z) \cong I(A)/I(A)^2 \cong A$, $H_1(A,I(U)) \cong I(U)/I(A)I(U)$ and $H_1(A,Z[U]) \cong Z[U]/I(A)Z[U] \cong Z[G]$. We identify these isomorphic groups, respectively. Then j is the restriction to I(U)/I(A)I(U) of the ring epimorphism $Z[U]/I(A)I(U) \longrightarrow Z[G]$ induced from the canonical epimorphism $U \rightarrow G$ of groups. Moreover, under the identification between $H_2(A,Z)$ and A, i is a monomorphism of the multiplicative group A to the additive group I(U)/I(A)I(U) defined by $i(a) = a-1 \mod I(A)I(U)$, $a \in A$.

Set B = I(U)/I(A)I(U). Then we have an exact sequence

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} I(G) \longrightarrow 0 \qquad (1)$$

of abelian groups. By multiplication in Z[U], U operates on B. Under this operation, A acts trivially as follows :

$$a(u-1) = (a-1)(u-1) + (u-1) \pmod{I(A)I(U)}, \text{ for } a \in A,\ u \in U.$$

Hence B can be viewed as a left G-module, and it is clear that j is a G-epimorphism.

Theorem. B is a splitting module for α .

Proof. Let $U = \bigcup_{x \in G} Au_x$ ($u_1 = 1$) be a decomposition of cosets of U with respect to A. Then the normalized 2-cocycle $a_{x,y}$ defined by $u_x u_y = a_{x,y} u_{xy}$ ($a_{x,y} \in A$) belongs to the given cohomology class α . Furthermore the conjugation $u_x a u_x^{-1}$ of $a \in A$ by u_x coincides with the original action xa on a by $x \in G$. Thus, for any $a \in A$ and $x \in G$ we have that

$$
\begin{aligned}
i(xa) &= u_x a u_x^{-1} - 1 \mod I(A)I(U) \\
&= u_x(a-1)(u_x^{-1}-1) + u_x(a-1) \mod I(A)I(U) \\
&= u_x(a-1) \mod I(A)I(U) = xi(a).
\end{aligned}
$$

This shows that i preserves the operations of G. On the other hand, for each $x \in G$ we define the element b_x of B by setting $b_x = u_x - 1 \mod I(A)I(U)$. Then , for any $x,y \in G$ it follows that

$$
\begin{aligned}
xb_y &= u_x(u_y-1) \mod I(A)I(U) \\
&= (a_{x,y}-1)(u_{xy}-1)+(a_{x,y}-1)+(u_{xy}-1)-(u_x-1) \mod I(A)I(U) \\
&= i(a_{x,y}) + b_{xy} - b_x , \qquad (2)
\end{aligned}
$$

which shows that the 2-cocycle $a_{x,y}$ becomes in B the coboundary of the 1-cochain b_x . This proves the theorem.

We see easily that our splitting module B is isomorphic to one constructed formally in Introduction.

2. Main properties of the module B.

Proposition 1. $B/i(A)$ is isomorphic to I(G) as G-modules.

Proof. This is a direct consequence of the exactness of the sequence (1) of G-modules.

Proposition 2. B/I(G)B is isomorphic to U/[U,U] as abelian groups, where [U,U] means the commutator of U.

Proof. By definition of the action on B of G, $I(G)B = I(U)^2/I(A)I(U)$. Then we have the isomorphisms $B/I(G)B \cong I(U)/I(U)^2 \cong U/[U,U]$.

In the following, we assume that G is finite.

Proposition 3. Let $c \in Z[G]$. Then $cB \subseteq i(A)$ if and only if

$$c = r \sum_{x \in G} x$$

for some integer $r \in Z$.

Proof. In the group ring Z[G] of a finite group G, we can see easily that $cI(G) = 0$ if and only if $c = r \sum_{x \in G} x$ for some $r \in Z$. Thus the proposition follows from Proposition 1.

Let t ; $B \longrightarrow B$ be the trace map, that is $t(b) = \sum_{x \in G} xb$. Since $t(I(G)B) = 0$, t induces a homomorphism $\bar{t}$; $B/I(G)B \longrightarrow B$.

Proposition 4. The following diagram commutes :

$$\begin{array}{ccc} U/[U,U] & \cong & B/I(G)B \\ \downarrow V_{U \to A} & & \downarrow \bar{t} \\ A & \xrightarrow{\ i\ } & B \end{array}$$

where $V_{U \to A}$ is the transfer map of groups.

Proof. In fact, from Proposition 3 the image of $\bar{t}$ is contained in $i(A)$. By definition of the transfer, $V_{U \to A}(au_y[U,U]) = \prod_x (xa) \prod_x a_{x,y}$, and so $i \circ V_{U \to A}(au_y[U,U]) = \sum_x xi(a) + \sum_x i(a_{x,y})$. On the other hand, the isomorphism $U/[U,U] \longrightarrow B/I(G)B$ sends the element $au_y[U,U]$ to $au_y - 1 + I(G)B$ which equals to the element $i(a) + b_y$, and from the equality of (2), it follows that $\bar{t}(i(a)+b_y) = \sum_x xi(a) + \sum_x i(a_{x,y})$. This shows that the above diagram is commutative.

Remark. These properties of the splitting module B are usefull in the group theoritical proof of the principal ideal theorem in class field theory (see, E.Weiss, ibid.).