On Skew Group Rings (Skew Polynomial Rings, Group Rings and Related Topics)

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Citation
数理解析研究所講究録 (1981), 438: 47-50

Issue Date
1981-09

URL
http://hdl.handle.net/2433/102780

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
On skew group rings

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Let $R$ be a ring with $1$, and $G$ a group. $U(R)$ denotes the group of units of $R$. Given maps $\alpha: G \to \text{Aut}(R)$ and $\gamma: G \times G \to U(R)$ such that

1. $\gamma(g,h)\gamma(gh,i) = \gamma(h,i)^{-1}\alpha(g)^{-1}\gamma(g,hi)$

and

2. $\gamma(g,h)r\alpha(gh)^{-1} = r\alpha(g)^{-1}\alpha(g)^{-1}\gamma(g,h)$

for all $g, h, i \in G$, $r \in R$, we define the crossed product $R^G$ to be a free $R$-module with basis $\{g | g \in G\}$. The multiplication is given by the rule

$$(rg)(rh) = r\gamma(g,h)^{-1}\gamma(g,h)\gamma(g,h)^{-1}.$$ 

This makes $R^G$ an associative ring with unit element $\gamma(1,1)^{-1}1$. The map $r \mapsto r\gamma(1,1)^{-1}1$ is a ring monomorphism of $R$ into $R^G$. We therefore consider $R$ as a subring of $R^G$.

If $\gamma(g,h) = 1$ for all $g, h \in G$, $R^G$ called a skew group ring, and if $\alpha(g) = 1$ for all $g \in G$, $R^G$ is called a twisted group ring.

Let $S$ be any ring. Let $R^G$ be a crossed product with $G$ a finite group, and $V, V'$ $(S,R^G)$-modules. For $g \in G$ and $k \in \text{Hom}(S,R)(V,V')$, we define $k^G(v) = k(vg^{-1})g$ for all $v \in V$. One may check that $k \mapsto k^G$ defines a group action of $G$ on $\text{Hom}(S,R)(V,V')$. It is clear that the fixed submodule is $\text{Hom}(S,R^G)(V,V')$. Therefore $t_g(k) = \sum_{g \in G} k^G$ is an $(S,R^G)$
omomorphism for every \( k \in \text{Hom}_{(S,R)}(V,V') \). If there exists an \( h \in \text{End}_{(S,R)}(V') \) such that \( t_G(h) = 1_{V'} \), then \( \hat{k} = t_G(hk) \) is an \((S,R^G)\)-homomorphism and \( \hat{k} = k \) on every \( R^G \)-submodule of \( V \).

\( C(R) \) denotes the center of \( R \). If there exists an element \( c \in C(R) \) such that \( t_G(c) = \sum_{g \in G} c^g = 1 \), then \( t_G(T_c) = 1 \), where \( T_c \in \text{End}_{(S,R)}(V') \) denotes right multiplication by \( c \).

If \( V' \) is \( |G| \)-torsion free and \( V \cdot |G| \), then we can define an element \( h \in \text{End}_{(S,R)}(V') \) by \( h(v) = |G|^{-1}v \) for all \( v \in V' \). Clearly, \( t_G(h) = 1 \).

Now, the proof of the following is easy.

Proposition 1. Let \( W \triangleleft V \) be \((S,R^G)\)-modules. Suppose there exists an element \( c \in C(R) \) such that \( t_G(c) = 1 \).

If \( W \triangleleft_{S^R} V_R \), then \( W \triangleleft_{S^R} V_{R^G} \).

We note that if the order of \( G \) is invertible in \( R \), then \( |G|^{-1} \in C(R) \) and \( t_G(|G|^{-1}) = 1 \).

A ring \( R \) is said to be fully right idempotent if every right ideal of \( R \) is idempotent. For example, von Neumann regular rings, right \( V \)-rings, and ring which are direct sum of simple rings, are fully right idempotent.

Corollary 1. Let \( R^G \) be a crossed product with \( G \) a finite group. Suppose there exists an element \( c \in C(R) \) such that \( t_G(c) = 1 \).

(1) If \( R \) is fully right idempotent, then so is \( R^G \).
(2) If $R$ is regular, then so is $R^*G$.

(3) If $R$ is a right V-ring, then so is $R^*G$.

(4) If $R$ is a direct sum of simple rings, then so is $R^*G$.

Proof. We prove only (3). If $K$ is a maximal right ideal of $R^*G$, then there exists a maximal submodule $M$ of $R^*G_R$ which contains $K$. It is easy to see that $\bigcap_{g \in G} Mg = K$. Therefore $P = R^*G/K$ is a direct sum of simple right $R$-modules, and hence $P$ is an injective $R$-module. Let $E$ be an injective hull of $P_{R^*G}$. Since $P \leq E_R$, $P = E$ by Proposition 1.

Let $G \subseteq \text{Aut}(R)$ be a finite group, and $R^*G$ a skew group ring. $R$ can be viewed as a right $R^*G$-module by defining $r \cdot \sum x_g g = \sum (rx_g)g$; for $x_g$ and $r$ in $R$. If we set $f = \sum_{g \in G} g$, then $R \cong fR^*G$ as right $R^*G$-modules. The fixed subring is denoted by $R^G$; $R^G = \{r \in R \mid r^g = r \text{ for all } g \in G\}$. For a module $V$ over a ring $S$, let $L(V_S)$ denote the lattice of $S$-submodules of $V$.

Lemma 1. Let $G \subseteq \text{Aut}(R)$ be finite. Suppose there is an element $c \in R$ such that $t_G(c) = 1$. Then the following are equivalent:

1) $R_{RfR}$ is s-unital; that is, $r \in r \cdot RfR$ for all $r \in R$.

2) $L(R^G_R) + L(R_{R^*G}) ; I + IR$, is a lattice isomorphism.

Proof. Since $t_G(r \cdot R^*G)R = r \cdot RfR$ for every $r \in R$, the assertion is clear.
Corollary 2. Let $R$ be a fully right idempotent ring, and $G \subseteq \text{Aut}(R)$ finite. Suppose there is an element $c \in C(R)$ such that $t_g(c) = 1$. Then the lattice of right ideals of $R^G$ is isomorphic to the lattice of $G$-invariant right ideals of $R$.

Proof. We set $Q = R^G$. By the part (1) of Corollary 1, $Q$ is fully right idempotent. Let $r$ be an element of $R$. Then, $fr \in (frQ)^2 \subseteq fr(RfR)$ and so $r \in R(RfR)$.

References


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