On skew group rings

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Let \( R \) be a ring with 1, and \( G \) a group. \( U(R) \) denotes the group of units of \( R \). Given maps \( \alpha : G \to \text{Aut}(R) \) and 
\( \gamma : G \times G \to U(R) \) such that

(i) \( \gamma(g,h)\gamma(gh,i) = \gamma(h,i)\alpha(g)^{-1}\gamma(g,h) \)

and

(ii) \( \gamma(g,h)\gamma(r\alpha(gh)^{-1}) = r\alpha(g)^{-1}\alpha(g)^{-1}\gamma(g,h) \)

for all \( g, h, i \in G, r \in R \), we define the crossed product \( R^G \)
to be a free \( R \)-module with basis \( \{ g^R \mid g \in G \} \). The multiplication is given by the rule

\( (r_g g) (r_h g) = r_g r_h \alpha(g)^{-1} \gamma(g,h) g^{gh} \).

This makes \( R^G \) an associative ring with unit element \( \gamma(l,l)^{-1}1 \). The map \( r \mapsto r\gamma(l,l)^{-1}1 \) is a ring monomorphism of \( R \) into \( R^G \). We therefore consider \( R \) as a subring of \( R^G \).

If \( \gamma(g,h) = 1 \) for all \( g, h \in G \), \( R^G \) called a skew group ring, and if \( \alpha(g) = 1 \) for all \( g \in G \), \( R^G \) is called a twisted group ring.

Let \( S \) be any ring. Let \( R^G \) be a crossed product with \( G \) a finite group, and \( V, V' \) \((S,R^G)\)-modules. For \( g \in G \) and \( k \in \text{Hom}(S,R)(V,V') \), we define \( k^g(v) = k(vg^{-1})g \) for all \( v \in V \). One may check that \( k \mapsto k^g \) defines a group action of \( G \) on \( \text{Hom}(S,R)(V,V') \). It is clear that the fixed submodule is \( \text{Hom}(S,R^G)(V,V') \). Therefore \( t_g(k) = \sum_{g \in G} k^g \) is an \((S,R^G)\)
omomorphism for every \( k \in \text{Hom}_{(S,R)}(V,V') \). If there exists an \( h \in \text{End}_{(S,R)}(V') \) such that \( t_G(h) = 1_{V'} \), then \( \hat{k} = t_G(hk) \) is an \((S,R^G)\)-homomorphism and \( \hat{k} = k \) on every \( R^G \)-submodule of \( V \). 

\( C(R) \) denotes the center of \( R \). If there exists an element \( c \in C(R) \) such that \( t_G(c) = \sum_{g \in G} c^g = 1 \), then \( t_G(T_c) = 1 \), where \( T_c \in \text{End}_{(S,R)}(V') \) denotes right multiplication by \( c \).

If \( V' \) is \(|G|\)-torsion free and \( V \cdot |G| \), then we can define an element \( h \in \text{End}_{(S,R)}(V') \) by \( h(v) = |G|^{-1}v \) for all \( v \in V' \). Clearly, \( t_G(h) = 1 \).

Now, the proof of the following is easy.

Proposition 1. Let \( W \subset V \) be \((S,R^G)\)-modules. Suppose there exists an element \( c \in C(R) \) such that \( t_G(c) = 1 \).

If \( W \underset{S}{\triangleleft} V_R \), then \( W \underset{S}{\triangleleft} V_R^G \).

We note that if the order of \( G \) is invertible in \( R \), then \( |G|^{-1} \in C(R) \) and \( t_G(|G|^{-1}) = 1 \).

A ring \( R \) is said to be fully right idempotent if every right ideal of \( R \) is idempotent. For example, von Neumann regular rings, right \( V \)-rings, and ring which are direct sum of simple rings, are fully right idempotent.

Corollary 1. Let \( R^G \) be a crossed product with \( G \) a finite group. Suppose there exists an element \( c \in C(R) \) such that \( t_G(c) = 1 \).

(1) If \( R \) is fully right idempotent, then so is \( R^G \).
(2) If $R$ is regular, then so is $R^*G$.
(3) If $R$ is a right $V$-ring, then so is $R^*G$.
(4) If $R$ is a direct sum of simple rings, then so is $R^*G$.

Proof. We prove only (3). If $K$ is a maximal right
ideal of $R^*G$, then there exists a maximal submodule $M$ of $R^*G_R$ which contains $K$. It is easy to see that $\bigcap_{g \in G} M_g = K$. Therefore $P = R^*G/K$ is a direct sum of simple right $R$-modules, and hence $P$ is an injective $R$-module. Let $E$ be an injective hull of $P_{R^*G}$. Since $P \triangleleft E_R$, $P = E$ by Proposition 1.

Let $G \leq Aut(R)$ be a finite group, and $R^*G$ a skew
group ring. $R$ can be viewed as a right $R^*G$-module by
defining $r \cdot \sum x_g g = \sum (rx_g)g$; for $x_g$ and $r$ in $R$. If we set $r = \sum_{g \in G} g$, then $R \simeq FR^*G$ as right $R^*G$-modules. The fixed
subring is denoted by $R^G$; $R^G = \{ r \in R \mid r^g = r \text{ for all } g \in G \}$.
For a module $V$ over a ring $S$, let $L(V_S)$ denote the lattice
of $S$-submodules of $V$.

Lemma 1. Let $G \leq Aut(R)$ be finite. Suppose there is
an element $c \in R$ such that $t_G(c) = 1$. Then the following
are equivalent:

1) $R_{RfR}$ is s-unital; that is, $r \in r \cdot RfR$ for all $r \in R$.
2) $L(R^G_{R^*G}) + L(R_{R^*G}); I \rightarrow IR$, is a lattice isomorphism.

Proof. Since $t_G(r \cdot R^*G)R = r \cdot RfR$ for every $r \in R$, the
assertion is clear.
Corollary 2. Let \( R \) be a fully right idempotent ring, and \( G \subset \text{Aut}(R) \) finite. Suppose there is an element \( c \in C(R) \) such that \( t_g(c) = c \). Then the lattice of right ideals of \( R^G \) is isomorphic to the lattice of \( G \)-invariant right ideals of \( R \).

Proof. We set \( Q = R^G \). By the part (1) of Corollary 1, \( Q \) is fully right idempotent. Let \( r \) be an element of \( R \). Then, \( fr \in (frQ)^2 \subseteq fr(RfR) \) and so \( r \in r \cdot (RfR) \).

References


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