A Characterization of QF-Rings (Skew Polynomial Rings, Group Rings and Related Topics)

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A Characterization of QF - rings

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It is well known that T. Nakayama found quasi-Probenius rings (QF-rings) as a generalization of group algebras [3]. Nowadays we know many characterizations of QF-rings. One of them is that R is self-injective as a right R-module. Other one is that every indecomposable and projective right ideal \( e_1R \) contains the unique minimal right ideal \( r_1 \) and if \( e_1R \neq e_2R, r_1 \neq r_2 \) and the above statements are valid for left ideals, where the \( e_i \) is a primitive idempotent.

In this note we shall combine the two weakened conditions. We always assume that \( R \) is a left and right artinian ring with identity and every \( R \)-module is a unitary right \( R \)-module.

Let \( M \) be an \( R \)-module. We consider a diagram for a minimal right ideal \( I \) of \( R \):

\[
\begin{array}{c}
0 \rightarrow I \xrightarrow{i} R \\
\downarrow{f} & h \\
M & \end{array}
\]

If there exists \( h \in \text{Hom}_R(R,M) \) with \( f = hi \) for any \( f \) and \( i \), we say \( M \) is right mini-injective. We can define similarly left mini-injective. If \( R \) is right mini-injective
as a right $R$-module, we say $R$ is right self mini-injective. We note that right self mini-injective ring is not left self mini-injective in general.

We shall show

Theorem 1. Let $R$ be a right and left artinian ring. Then $R$ is QF-ring if and only if $R$ is right and left self mini-injective.

Theorem 2. Let $R$ be an algebra over a field $K$ of finite dimension. Then $R$ is a QF-algebra if and only if $R$ is a right self mini-injective algebra.

Remark. Theorem 2 is not valid for artinian rings.

Proofs of Theorems.

We shall publish the results in this note in [1] and [2] and so we give here a sketch of the proofs.

In this note, we consider only artinian rings and so from now on we understand a ring $R$ is always right artinian. We denote the Jacobson radical by $J = J(R)$.

We put $\overline{R} = R/J$. Let

$$R = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{p(i)} e_{ij} R$$

be the standard decomposition, namely the $e_{ij}$ is a primitive idempotent and $e_{ij} R = e_{ii} R$, $e_{ii} R \neq e_{jj} R$ if $i \neq j$. If the socle $S(e_{ii} R)$ of $e_{ii} R$ is simple for each $i$, then we say $R$ is right QF-2.
Proposition 1. Let \( R \) be as above. If \( R \) is right self mini-injective, then

1) If \( e_1R \neq e_2R \), each minimal submodule in \( e_1R \) is not isomorphic to any minimal one in \( e_2R \).

2) \( S(e_1R) = e_1J^k \) and every minimal submodule in \( e_1R \) is isomorphic each other.

3) \( r(J) \supseteq 1(J) \).

where the \( e_i \) is a primitive idempotent, \( r(J) = \{x \in R \mid Jx = 0\} \) and \( 1(J) = \{x \in R \mid xJ = 0\} \).

Proof. 1) is clear from the definition.

2) We take a minimal right ideal \( I \) in \( e_1J^k \neq 0 \) (\( e_1J^{k+1} = 0 \)). Using the definition and 1), we can show \( S(e_1R) = e_1J^k \).

3) Let \( x \) be in a minimal right ideal in \( e_1R \). Then we can show \( Jx \subseteq \bigcap_{i=1}^{\infty} e_1Jx = 0 \) from 1) and 2), where \( 1 = \bigcap_{i=1}^{\infty} e_i \).

Proof of Theorem 2.

First we shall show that \( R \) is right QF-2. Let \( I_1 \) be a minimal right ideal in \( e_1R \). We assume \( I_1 = \frac{e_2}{e_2}R \). Then we can show from the definition and Proposition 1 that there is a monomorphism of \( \text{End}_R(e_2R) \) to \( \text{End}_R(e_1R) \). Hence, \([\frac{e_1Re_1}{K}] = [\frac{e_2Re_2}{K}]\). Next take a minimal right ideal \( I_2 \) in \( e_2R \). Repeating this argument, we have a chain \( \{e_1, e_2', \ldots\} \) of primitive idempotents. Then \( e_1 = et' \) for some \( t \) by Proposition 1. Hence,
\[ [e_1 R e_1^*: K] = [e_2^T R e_2^T: K] \] implies that \( I_1 \) is a unique minimal right ideal in \( e_1 R \). Accordingly \( e_1 R \) is uniform and so an injective envelope \( E(e_1 R) \) of \( e_1 R \) is indecomposable. Therefore, \( E(e_1 R) \cong \text{Hom}_K(Re_1^*(i), K) \). Since \( S(e_1 R) \not\cong S(e_j R) \) if \( e_1 R \neq e_j R \) by Proposition 1, \( [R; K] = \bigoplus_{i=1}^{n} [Re_1^*(i); K] = \bigoplus_{i=1}^{n} [E(e_1 R); K] = [E(R); K] \).

Proof of Theorem 1.

Let \( xR \) be a minimal right ideal in \( e_1 R \) and \( xR = e_2^T R \). Since \( Jx = 0 \) by Proposition 1 and \( R \) is left self mini-injective, for any element \( b \) in \( e_1 R e_1^* \), there exists \( a \) in \( e_2^T R e_2^T \) such that \( bx = xa \) as above. Hence, \( xR = S(e_1 R) \) and \( R \) is left and right QF-2.

References


-4-