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A type of strongly radical polynomials

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Throughout the present note, \( R \) will represent a commutative algebra over \( GF(p) \). Unadorned \( \otimes \) means \( \otimes_R \), every module is \( R \)-module and every map is \( R \)-linear. Given an element \( u \) in \( R \), we denote by \( H_u \) the free Hopf algebra over \( R \) with basis \( \{1, \delta, \ldots, \delta^{p-1}\} \) whose Hopf algebra structure is given by

\[
\delta^p = 0, \quad \Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta + u(\delta \otimes \delta), \quad \Delta(\delta^j) = \Delta(\delta)^j, \\
\varepsilon(\delta) = 0, \quad \varepsilon(\delta^j) = \varepsilon(\delta)^j, \quad \lambda(\delta) = \sum_{i=1}^{p-1} (-1)^{i} u^{i-1} \delta^i \quad \text{and} \\
\lambda(\delta^j) = \lambda(\delta)^j \quad (1 \leq j \leq p - 1),
\]

where \( \Delta, \varepsilon \) and \( \lambda \) are the comultiplication, counit and antipode of \( H_u \), respectively.

In this note we study on quadratic extension and \( H_u \)-Hopf Galois extension of \( R \).

Let \( A \) be a commutative \( R \)-algebra and \( \mu : A \otimes A \rightarrow A \) a multiplication map. \( A \) is called a purely inseparable algebra in the sense of Sweedler if \( \text{Ker}(\mu) \leq J(A \otimes A) \), the Jacobson radical of \( A \otimes A \) ([5, Def.1 and Lemma 1 (a)]). \( A \) is called a strongly radical if \( A \) is f. g. projective \( R \)-module and \( \text{Ker}(\mu) \) is nilpotent.

First, we have the following
Theorem 1. Let \( A = R[X]/(X^2 - rX - s) \) \((r, s \in R)\). Assume \( p = 2 \). Then

1. \( A \) is purely inseparable if and only if \( r \in J(R) \).
2. \( A \) is strongly radicial if and only if \( r \) is nilpotent.

Proof. Noting that \( \text{Ker}(\mu) \) is generated by \( y = x \otimes 1 + 1 \otimes x \) as \( A \)-module and \( y^2 = ry \), (2) is clear. Thus we prove (1).

If \( r \in J(R) \) then \( r \in J(A \otimes A) \), since \( A \otimes A \) is integral over \( R \). Thus \( y^2 = ry \in J(A \otimes A) \), whence it follows that \( y \in J(A \otimes A) \).

Let \( y \in \text{Ker}(\mu) \subseteq J(A \otimes A) \). Then \( 1 + cy \) is invertible for any \( c \in R \). Let \( z = t_0 + t_1(x \otimes 1) + t_2(1 \otimes x) + t_3(x \otimes x) \) be the inverse element of \( 1 + cy \) \((t_i \in R)\). Then we obtain

\[
\begin{bmatrix}
1 & cs & cs & 0 \\
c & 1 + cr & 0 & cs \\
c & 0 & 1 + cr & cs \\
0 & c & c & 0
\end{bmatrix}
\begin{bmatrix}
t_0 \\
t_1 \\
t_2 \\
t_3
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

As is easily seen

\[
(1 + cr)^2 \begin{bmatrix}
t_0 \\
t_1 \\
t_2 \\
t_3
\end{bmatrix} = (1 + cr) \begin{bmatrix}
1 + cr \\
c \\
c \\
0
\end{bmatrix}.
\]

Then, by the uniqueness of the inverse of \( 1 + cy \), the matrix of the coefficients of \( t_i \) is invertible, and so the determinant of it is a nonzero divisor ([2, p.161, Cor.]). We have thus the following
For any $c \in R$, there exists $t \in R$ such that
\[(*) \quad (1 + cr)t = c\]

If $r \in J(R)$, then there exists a maximal ideal $M$ in $R$ such that $R = Rr + M$. Put $1 = r_0r + m \ (r_0 \in R, \ m \in M)$. Then by $(*)$, there exists $t \in R$ such that $(1 + r_0r)t = r_0$. Thus $r_0 = (1 + r_0r)t = mt \in M$. This implies a contradiction $1 \in M$. Hence $r \in J(R)$.

Remark 2. Let $A := R[X]/(x^2 - rx - s)$. Assume 2 is invertible in $R$. Then we can show the following:

1. $A$ is purely inseparable if and only if $r^2 + 4s \in J(R)$.
2. $A$ is strongly radical if and only if $r^2 + 4s$ is nilpotent.

Now, we consider $H_u$-Hopf Galois extension of $R$.

An $R$-algebra $A$ is called a projective $R$-algebra if $A$ is a projective $R$-module and $R$ is an $R$-direct summand of $A$. An $R$-algebra $A$ is called an $H_u$-module algebra if $A$ is an $H_u$-module such that the followings hold: For any $a, b \in A$,
\[
\delta(ab) = a\delta(b) + \delta(a)b + u\delta(a)\delta(b) \quad \text{and} \quad \delta(1) = 0.
\]

For an $H_u$-module algebra $A$, the smash product $A \# H_u$ is equal to $A \otimes H_u$ as an $R$-module but with multiplication
\[
(a \# h)(b \# k) = \sum (h) a(h(1)b) \# h(2)k
\]
where $\Delta(h) = \sum (h) h(1) \otimes h(2)$. In our case we have
\[
(1 \# \delta)(a \# 1) = \delta(a) \# 1 + a \# \delta + u\delta(a) \# \delta.
\]

A commutative $R$-algebra $A$ is called an $H_u$-Hopf Galois extension.
of $R$ if $A$ is a f. g. projective $H_u$-module algebra and the map
$\phi: A \# H_u \to \text{Hom}_R(A, A)$ defined by $\phi(a \# h)(x) = ah(x)$ is an $R$-
alg isomorphism. Note that $A$ is an $H_u$-Hopf Galois extension
of $R$ if and only if $A$ is a Galois $H^x = \text{Hom}_R(H, R)$-object in
the sense of Chase-Sweedler ([1, Th.9.3]).

**Theorem 3** ([3, Cor.1.6]). Let $A$ be a f. g. projective $H_u$
module algebra. Assume $p = 2$. Then the followings are
equivalent.

1. $A$ is an $H_u$-Hopf Galois extension of $R$.

2. There exists an element $x \in A$ such that $\delta(x)$ is
invertible in $R$ and $x^2 = ux + s$ for some $s \in R$. When this
is the case, $A$ is a free $R$-module with basis \{1, x\}.

This theorem is generalized as follows.

**Theorem 4** ([4]). If $A$ is an $H_u$-Hopf Galois extension of $R$, then there exists $x \in A$ such that $\delta(x) = 1$ and $x^p = u^{p-1}x + r_0$ for some $r_0 \in R$. When this is the case \{1, x, ..., x^{p-1}\}
is a free basis of $A$. Conversely, let $f(X) = x^p - r_1X - r_0 \in R[X]$. If there exists $v \in R$ such that $v^{p-1} = r_1$, then $A = R[X]/(f(X))$ is an $H_v$-Hopf Galois extension of $R$.

**Remark 5.** In Th.4, if $u$ is nilpotent, then $A$ is purely
inseparable, and if $u = 1$, then $f(X)$ is an Artin-Schreier
polynomial. In detail, see [4].
These extensions are $p$-extensions. We give a simple example of $p^m$-extension. Assume $p = 2$. Let $H$ be a Hopf algebra with free basis $\{1, \delta, \delta^2, \delta^3\}$ such that the Hopf algebra structure is defined by

$$
\delta^4 = 0, \quad \Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta + u(\delta \otimes \delta), \quad \Delta(\delta^j) = \Delta(\delta)^j,
$$

$$
\epsilon(\delta) = 0, \quad \epsilon(\delta^j) = \epsilon(\delta)^j, \quad \lambda(\delta) = \sum_{i=1}^{3} (-1)^i u^{i-1} \delta^i \quad \text{and}
$$

$$
\lambda(\delta^j) = \lambda(\delta)^j \quad (1 \leq j \leq 3).
$$

Since $H$ is a Galois $H$-object with comodule structure map $\Delta$: $H \rightarrow H \otimes H$ ([1, Prop.9.1]), $H$ has an $H^*$-module algebra structure defined by $h^* \mapsto h = \sum (h)^* (h(1))^* h(2)$. Thus $H$ is an $H^*$-Hopf Galois extension of $R$. Replacing $H$ with $H^*$, $H^*$ is an $H$-Hopf Galois extension of $R$. Using the $H^*$-module structure, it can be seen that

$$
H^* \cong R[X]/(X^2 - uX) \otimes R[Y]/(Y^2 - uY)
$$

as $H$-Hopf Galois extension, where $X, Y$ are indeterminates. This extension is not isomorphic to cyclic extension. A Hopf algebra which corresponds to a cyclic extension is not $H_u$-type.
References


