<table>
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<th>Realization of Lie Algebras (Skew Polynomial Rings, Group Rings and Related Topics)</th>
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<tbody>
<tr>
<td>Author(s)</td>
<td>MORIKAWA, HISASHI</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 438: 6-8</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1981-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/102788">http://hdl.handle.net/2433/102788</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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REALIZATIONS OF LIE ALGEBRAS

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This is an expository talk on realizations of Lie algebras.

1. The classical invariant theory

Let us choose a generic polynomial

\[ f(\xi | z) = \sum_{k=0}^{n} \xi_z^{(k)}(z), \]

on which \( SL_2(K) \) acts as follows

\[ f(\begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \xi | z) = (\gamma z + \delta)^{m} f(\xi | \gamma z + \delta), \]

i.e.,

\[ \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \xi_z^{(k)}(z) = \sum_{p,q} \xi_{p,q}^{(k)}(z) \xi_z^{(k-p+q)}(z) = p,q,n-k-q. \]

The corresponding realization of \( sl_2(K) \) is given by

\[ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rightarrow \Delta = \sum_{\xi} (n-\xi) \xi_z^{(k+1)} \frac{\partial}{\partial \xi_z^{(k)}}, \]

\[ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \Delta = \sum_{\xi} (n-k) \xi_z^{(k)} \frac{\partial}{\partial \xi_z^{(k)}}, \]

\[ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \Delta = \sum_{\xi} (n-2k) \xi_z^{(k)} \frac{\partial}{\partial \xi_z^{(k)}}. \]

Definition 1.

\[ \rho [m] = \{ \text{covariants of index } m \} \]
\[
F(\xi, z) = \sum_{\ell} \left( \sum_{m} c_{\ell}^{(m)} (\xi) z^{\ell} \right) c_{\ell}^{(z)} (z) \in K[\xi],
\]

\[
F\left( \frac{\delta}{\gamma} \frac{\beta}{\alpha} \xi, z \right) = (\gamma z + \delta)^m F(\xi, \alpha z + \beta).
\]

**Definition 2.**

\[\mathcal{G}^{[m]} = \{\text{semi-invariants of index } m\}\]

\[= \{\phi \in K[\xi] | \Theta \phi = 0, \Upsilon \phi = m \phi\} \]

**Problem.** To seek all covariants of index \(m\).

**Solution.** (Robert's theorem)

\[\rho^{[m]} = \{\exp(z \Delta) \phi(\xi) | \phi(\xi) \in \mathcal{G}^{[m]}\} \]

**Remark.** This solution \(\exp(z \Delta) \phi(\xi)\) is a typical explicit solution of mathematical problems.

2. **Automorphic forms**

Let us choose a formal power series

\[f(\xi | z) = \sum_{\ell=0}^{\infty} \frac{(-2k)^{\ell}}{\ell!} \xi^{(\ell)} z^{\ell}\]

with variable coefficients, where

\[(-2k)^{\ell} = (-2k)(-2k-1)\cdots(-2k-\ell+1)\]

Denoting

\[
\Theta = \sum \xi^{(\ell-1)} \frac{\partial}{\partial \xi^{(\ell)}} ,
\]

\[
\Lambda = \sum (-2k-\ell) \xi^{(\ell+1)} \frac{\partial}{\partial \xi^{(\ell)}} ,
\]

\[
\Upsilon = \sum (-2k-2\ell) \xi^{(\ell)} \frac{\partial}{\partial \xi^{(\ell)}} ,
\]

we have a realization of \(sl_2(\mathbb{C})\). Denote
\( \mathcal{G}[-2m] = \{ \varphi \in \mathbb{k}[\xi] | \mathcal{O}\varphi = 0, \mathcal{M}\varphi = -2m\varphi \} \).

**Problem.** Let \( h(z) \) be an automorphic form of dimension \(-2k\). To seek all automorphic forms of dimension \(-2m\) which are differential polynomials of \( h(z) \).

**Solution.** Assume that the Zariski closure of the automorphic group coincides with \( \text{PSL}_2(\mathbb{C}) \). And denote

\[
h(z) = \sum_{\ell=0}^{\infty} \frac{(-2k)^{\ell}}{\ell!} a^{(\ell)} z^\ell.
\]

Then

\[
\{ \exp(z\Delta)\varphi(\xi) |_{\xi=a} | \varphi(\xi) \in \mathcal{G}[-2m] \}
\]

\[
= \left\{ \text{automorphic forms of dimension } -2m \right\}
\]

\[
\left\{ \text{which are differential polynomials of } h(z) \right\}.
\]

**Reference**

Hisasi Morikawa, Invariant theory, Kinokuniya.