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REALIZATIONS OF LIE ALGEBRAS

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This is an expository talk on realizations of Lie algebras.

1. The classical invariant theory

Let us choose a generic polynomial

\[ f(\xi|z) = \sum_{k=0}^{n} \left( \frac{n}{k} \right) \xi_{k}^{(k)} z^{(k)} , \]

on which \( SL_{2}(K) \) acts as follows

\[ f\left( \delta \beta \right) = (\gamma z + \delta)^{m} f\left( \xi_{\gamma z + \delta} \right) , \]

i.e.,

\[ \left( \delta \beta \right) \xi_{\gamma}(z) = \sum_{p,q} \left( \frac{k}{p} \right) \left( \frac{n-k}{q} \right) \left( \frac{z-p+q}{\xi_{-p+q}} \right) \left( \frac{z-p-q}{\xi_{-p-q}} \right) . \]

The corresponding realization of \( sl_{2}(K) \) is given by

\[
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\rightarrow \mathcal{B} = \sum_{k} \xi_{k}^{(k-1)} \frac{a}{\delta \xi(k)} ,
\]

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\rightarrow \Delta = \sum_{k} \xi_{n-k}^{(k+1)} \frac{a}{\delta \xi(k)} ,
\]

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\rightarrow \mathcal{M} = \sum_{k} \xi_{n-2k}^{(k)} \frac{a}{\delta \xi(k)} .
\]

Definition 1.

\[ p^{[m]} = \{ \text{covariants of index } m \} \]
\[ F(\xi, z) = \sum_{k=0}^{m} c_k(\xi) z^k \in K[\xi], \]
\[ F(\delta \beta) F(\gamma \alpha) F(\xi, z) = (\gamma z + \delta)^m F(\xi, \alpha z + \beta) \].

Definition 2.
\[ \mathcal{G}^{[m]} = \{ \text{semi-invariants of index } m \} \]
\[ = \{ \phi \in K[\xi] | \delta \phi = 0, \mu \phi = m \phi \}. \]

Problem. To seek all covariants of index \( m \).

Solution. (Robert's theorem)
\[ \rho^{[m]} = \{ \exp(z \Lambda) \phi(\xi) | \phi(\xi) \in \mathcal{G}^{[m]} \}. \]

Remark. This solution \( \exp(z \Lambda) \phi(\xi) \) is a typical explicit solution of mathematical problems.

2. Automorphic forms

Let us choose a formal power series
\[ f(\xi | z) = \sum_{k=0}^{\infty} \frac{(-2k)_k}{k!} \xi(\xi) z^k, \]
with variable coefficients, where
\[ (-2k)_k = (-2k)(-2k-1)\cdots(-2k-k+1). \]

Denoting
\[ \Theta = \sum_{k=0}^{\infty} \xi(\xi) \frac{\partial}{\partial \xi}, \]
\[ \Delta = \sum_{k=-1}^{\infty} (-2k-k) \xi(\xi+1) \frac{\partial}{\partial \xi}, \]
\[ \mu = \sum_{k=2}^{\infty} (-2k-2k) \xi(\xi) \frac{\partial}{\partial \xi}, \]
we have a realization of \( sl_2(\mathbb{C}) \). Denote
\[ \mathcal{G}[-2m] = \{ \phi \in K[\xi] | \mathcal{O}\phi = 0, \mathcal{M}\phi = -2m\phi \}. \]

Problem. Let \( h(z) \) be an automorphic form of dimension \(-2k\). To seek all automorphic forms of dimension \(-2m\) which are differential polynomials of \( h(z) \).

Solution. Assume that the Zariski closure of the automorphic group coincides with \( \text{PSL}_2(\mathbb{C}) \). And denote

\[ h(z) = \sum_{\ell=0}^{\infty} \frac{(-2k)^{\ell}}{\ell!} \alpha(\xi) z^\ell. \]

Then

\[ \{ \exp(z\Delta)\phi(\xi) | \xi = \alpha | \phi(\xi) \in \mathcal{G}[-2m] \} = \left\{ \text{automorphic forms of dimension } -2m \right\} \]

\[ \left\{ \text{which are differential polynomials of } h(z) \right\}. \]

Reference
Hisasi Morikawa, Invariant theory, Kinokuniya.