REALIZATIONS OF LIE ALGEBRAS

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This is an expository talk on realizations of Lie algebras.

1. The classical invariant theory

Let us choose a generic polynomial

\[ f(\xi | z) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \xi_{z}^{(k)} , \]

on which \( SL_2(K) \) acts as follows

\[ f(\left( \begin{array}{c} \delta \\ \gamma \\ \alpha \\ \beta \end{array} \right) \xi | z) = (\gamma z + \delta)^{m} f(\xi | \gamma z + \delta), \]

i.e.,

\[ \left( \begin{array}{c} \delta \\ \gamma \\ \alpha \\ \beta \end{array} \right) \xi_{z}^{(k)} = \sum_{p,q} \left( \begin{array}{c} n \end{array} \right)_{k} \xi_{z}^{(k-p+q)} \left( \begin{array}{c} n \end{array} \right)_{p} \left( \begin{array}{c} n \end{array} \right)_{q} \]

The corresponding realization of \( sl_2(K) \) is given by

\[
\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \mathcal{O} = \sum \xi_{z}^{(k-1)} \frac{a}{\partial \xi^{(k)}}, \\
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \mathcal{A} = \sum (n-k) \xi_{z}^{(k+1)} \frac{a}{\partial \xi^{(k)}}, \\
\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \mathcal{N} = \sum (n-2k) \xi_{z}^{(k+1)} \frac{a}{\partial \xi^{(k)}}.
\]

Definition 1.

\[ \rho^{[m]} = \{ \text{covariants of index} \ m \} \]
\[
F(\xi, z) = \sum_{\ell} \left\{ \begin{array}{l}
\xi(\xi) z^\ell \in K[\xi], \\
F(\frac{\delta}{\gamma} \frac{\beta}{\alpha} \xi, z) = (\gamma z + \delta)^m F(\xi, \alpha z + \beta)
\end{array} \right. 
\]

Definition 2.

\[\mathcal{G}^{[m]} = \{ \text{semi-invariants of index } m \} = \{ \phi \in K[\xi] \mid \Theta \phi = 0, \mathcal{M} \phi = m \phi \}.\]

Problem. To seek all covariants of index \( m \).

Solution. (Robert's theorem)

\[\rho^{[m]} = \{ \exp(z \Delta) \phi(\xi) \mid \phi(\xi) \in \mathcal{G}^{[m]} \}.\]

Remark. This solution \( \exp(z \Delta) \phi(\xi) \) is a typical explicit solution of mathematical problems.

2. Automorphic forms

Let us choose a formal power series

\[f(\xi|z) = \lim_{\ell \to \infty} \frac{(-2k)^\ell}{\ell!} \xi(\xi) z^\ell\]

with variable coefficients, where

\[(-2k)^\ell = (-2k)(-2k-1) \cdots (-2k-\ell+1).\]

Denoting

\[\Theta = \sum \xi(\xi)^{\ell-1} \frac{\partial}{\partial \xi(\xi)}, \]

\[\Delta = \sum (-2k-\ell) \xi(\xi)^{\ell+1} \frac{\partial}{\partial \xi(\xi)} ,\]

\[\mathcal{M} = \sum (-2k-2\ell) \xi(\xi) \frac{\partial}{\partial \xi(\xi)},\]

we have a realization of \( \text{sl}_2(\mathbb{C}) \). Denote
\[ G[-2m] = \{ \varphi \in K[\xi] | \mathcal{O}\varphi = 0, \mathcal{H}\varphi = -2m\varphi \} . \]

Problem. Let \( h(z) \) be an automorphic form of dimension \(-2k\). To seek all automorphic forms of dimension \(-2m\) which are differential polynomials of \( h(z) \).

Solution. Assume that the Zariski closure of the automorphic group coincides with \( \text{PSL}_2(\mathbb{C}) \). And denote

\[ h(z) = \sum_{\ell=0}^{\infty} \frac{(-2k)^\ell}{\ell!} a(\xi) z^\ell . \]

Then

\[ \{ \exp(zA)\varphi(\xi) \mid \xi = a, \varphi(\xi) \in G[-2m] \} \]

\[ = \left\{ \text{automorphic forms of dimension } -2m \text{ which are differential polynomials of } h(z) \right\} . \]

Reference

Hisasi Morikawa, Invariant theory, Kinokuniya.