Topics on Kac-Moody Lie algebras

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In this note we introduce Kac-Moody Lie algebras, their representation theory, and relation with theory of differential equations.

I. Kac-Moody Lie algebras ([6],[14],[15])

Let \( C=(c_{ij}) \) be a \( n \times n \)-matrix satisfying following conditions;

\[ c_{ij} \in \mathbb{Z}, \quad c_{ii} = 2, \quad c_{ij} < 0 \quad (i \neq j), \quad c_{ij} = 0 \iff c_{ji} = 0, \]

and define Lie algebra \( \mathfrak{L}(C) \) over \( \mathbb{C} \) with generators \( \{ h_i, e_i, f_i \mid i=1, \ldots, n \} \) and relations;

\[
\begin{align*}
[h_i, h_j] &= 0, \quad [h_i, e_j] = c_{ij} e_j, \quad [h_i, f_j] = -c_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_i, \\
(ad e_i)(-c_{ij} + I) e_j &= 0, \quad (ad f_i)(-c_{ij} + I) f_j = 0.
\end{align*}
\]

\( \mathfrak{L} = \mathfrak{L}(C) \) is called Kac-Moody Lie algebra.

\( \mathfrak{L} \) has \( \bigwedge^i \mathbb{Z}^n \) grading with \( \deg h_i = (0, \ldots, 0) \), \( \deg e_i = (0, \ldots, 1, \ldots, 0) = a_i \), and \( \deg f_i = -a_i \).

Put \( \mathfrak{L}_a = \{ x \in \mathfrak{L} \mid \deg x = a \} \), \( m_a = \dim \mathfrak{L}_a \), \( \Delta = \{ a \in \mathbb{N} \mid m_a \neq 0 \} \).

From defining relations \( \mathfrak{L} \) has vector space decomposition

\[ \mathfrak{L} = \bigoplus \mathfrak{L}_a, \quad \mathfrak{L}_a = \bigoplus \mathfrak{L}_i, \quad \mathfrak{L}_i = \langle e_1, \ldots, e_n \rangle, \quad \mathfrak{L}_i = \langle f_1, \ldots, f_n \rangle. \]

So \( \Delta \) is disjoint union of \( \Delta_+ = \bigcap_+ \cap \Delta \) and \( \Delta_- = (-\bigcap_+) \cap \Delta \)

\( \bigcap_+ = \mathbb{Z} \cdot a_1 + \cdots + \mathbb{Z} \cdot a_n \)

For each \( i \), define \( s_i \) \( GL(n) \) by

\[ s_i(a_j) = a_j - c_{ij} a_i \]

and \( W = \langle s_i \mid i=1, \ldots, n \rangle \).

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If $\mathbf{C}$ is decomposable, i.e. has a permutation matrix $P$ such that 

$$
\mathbf{PCP}^{-1} = \begin{pmatrix}
\mathbf{C}_1 & 0 \\
0 & \mathbf{C}_2
\end{pmatrix},
$$

then $\mathcal{L}(\mathbf{C}) = \mathcal{L}(\mathbf{C}_1) + \mathcal{L}(\mathbf{C}_2)$, $W(\mathbf{C}) = W(\mathbf{C}_1) \times W(\mathbf{C}_2)$. 

So we assume $\mathbf{C}$ is indecomposable.

A finite-dimensional complex simple Lie algebra is isomorphic to a $\mathcal{L}(\mathbf{C})$ whose $W(\mathbf{C})$ is finite group. Equivalent condition on $\mathbf{C}$ to finiteness of $W(\mathbf{C})$ is that there exist positive numbers $d_1, \ldots, d_n$ such that 

$$
(\begin{array}{cccc}
d_1 & & \\
& \ddots & \\
& & d_n
\end{array}) \mathbf{C} \text{ is a positive definite symmetric matrix.}
$$

If we replace "positive definite" by "positive semi-definite" in this condition, we get a new class of infinite-dimensional Lie algebras. They are called Euclidean (or affine) Lie algebras. We deal with Euclidean case hereafter.

2. Character formula and denominator formula

Put $V = \mathbb{C} \otimes \bigodot \mathfrak{F}$, and for $\lambda \in \mathfrak{F}^+$ let $c_\lambda : W \rightarrow V$ be the I-cocycle, i.e. satisfies 

$$
c_\lambda (\mathbf{w}_1 \mathbf{w}_2) = c_\lambda (\mathbf{w}_1) c_\lambda (\mathbf{w}_2), \text{ such that } c_\lambda (h_i) = \lambda (h_i) a_i \text{ (i=1, \ldots, n)}. 
$$

If $\lambda (h_i) \in \mathbb{Z}$ for all $i$, $\lambda$ is said integral, and moreover if $\lambda (h_i) \geq 0$, $\lambda$ is said dominant integral. It is easy to see 

$$
\begin{align*}
\lambda \text{: integral} & \Rightarrow c_\lambda (W) \subset \mathfrak{F} \\
\lambda \text{: dominant integral} & \Rightarrow c_\lambda (W) \subset \mathfrak{F}^+
\end{align*}
$$

To each dominant integral, there exists $\mathcal{L}$-module $V^\lambda$, unique up to isomorphism, with the following property:

$$
0 \neq v \in V \text{ s.t. } \begin{cases}
eq v = 0, & h_i v = \lambda (h_i) v, & f_i \lambda (h_i) v = 0 \text{ (i=1, \ldots, n)}
\end{cases}
$$

$$
V = \sum c_{f_i} f_i v
$$

$V^\lambda$ is called standard module with highest weight $\lambda$.

For $a \in \mathfrak{F}^+$, let 

$$
V^\lambda_a = \sum c_{f_i} f_i v
$$

and put 

$$
\text{ch } V^\lambda_a = \sum_{\alpha \in \mathfrak{F}^+} \lambda_{\alpha} (-a) \cdot a_{i_1} \cdots a_{i_k}
$$

(ch $V^\lambda$ is in $\mathbb{Z} [[(-a_1), \ldots, e(-a_n)]]$. )
Let \( \rho \in \mathfrak{g} \) be defined by \( \rho(h_i) = I \) for all \( i \). We have

character formula \( ([5],[8],[10]) \)

\[
\text{ch } V^\chi = \frac{\sum_{w \in W} \det(w) e(-c_{\Lambda +} \rho(w))}{\sum_{w \in W} \det(w) e(-c_\rho(w))}
\]

and

denominator formula

\[
\sum_{w \in W} \det(w) e(-c_{\Lambda +} \rho(w)) = \prod_{a \in \Delta^+} (1 - e(-a))^m_a
\]

If we make specialization \( e(-a_i) \rightarrow \frac{1}{q^{m_i}} \) \((m_1, \ldots, m_n \geq 0)\) to both sides of

denominator (or character ) formula, we have various combinatorial identities. \( ([7],[8],[12],[16]) \)

3. K-dV equation and construction of representation

Let \( u(x), V_{n1}(x) \ (n=1, 2, 3, \ldots, 1 \leq k \leq 2n-2) \) be functions in infinite many variables \( x=(x_1, x_2, x_3, \ldots) \). From the compatibility conditions of linear partial differential equations

\[
\begin{cases}
\left( \frac{\partial^2}{\partial x_1^2} \right) u(x) + \lambda u(x) = \lambda y(x) \\
\frac{\partial^{2n-2}}{\partial x_1^{2n-2}} u(x) + V_{n1} \frac{\partial^{2n-1}}{\partial x_1^{2n-1}} u(x) + \cdots + V_n \frac{\partial^{2n-2}}{\partial x_1^{2n-2}} u(x) = y(x) \quad (n=1, 2, \ldots)
\end{cases}
\]

we obtain non-linear differential equations on \( u \), and \( V_{n1} \) are expressed by differential polynomials of \( u \). These differential equations are called K-dV equations. A function \( \zeta(x) \) is called \( \zeta \)-function, if \( u = \frac{2}{\partial \zeta} \log \zeta \) is a solution of K-dV equations. The space of infinitesimal trasformations of \( \zeta \)-functions forms a Lie algebra \( \mathfrak{L} \) of linear trans-formations on \( \mathbb{C}[x_1, x_2, \ldots] \). Date, Jimbo, Kashiwara and Miwa find \( ([1]) \) that \( \mathfrak{L} \) is \( \mathbb{L} \)-module \( \mathbb{C}[x_1, x_2, \ldots] \) coincides with the standard module constructed by Lepowsky and Wilson. \( ([1]) \). Kac, Kazhdan, Lepowsky and Wilson construct standard modules for other Euclidean Lie algebras in analogous way to \( \mathbb{L} \), and D-J-K-M show
they correspond to some non-linear equations like K-dV.[2],[3],[9]

4. Remarks

(1) About classification and realizations of Euclidean Lie algebras, c.f. [6],[15].

(2) Let \( \mathfrak{g}(C) \) be the maximam homogenous ideal with \( \mathfrak{g} \cap \mathfrak{s} = 0 \). Strictly speaking, \( \mathfrak{g}(C) = \mathfrak{L}(C)/\mathfrak{R} \) is Kac-Moody Lie algebra. But finite or Euclidean case \( \mathfrak{R} = 0 \), and we conjecture that \( \mathfrak{R} = 0 \) for any case.

(3) Character formula and denominator formula are analogy of finite case, and proved for \( \mathfrak{g}(C) \) with symmetrizable \( C \), i.e. there exist positive numbers \( d_1, \ldots, d_n \) such that \( \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} C \) is symmetric matrix.

(4) Original K-dV equation is obtained by to put \( x_s, x_\tau, \ldots \) constant.

(5) Frenkel and Kac ([4]) construct standard modules in different way from \( K-K-L-W \). It will be interesting to describe isomorphism explicitly between \( K-K-L-W \)'s module and F-K's.

References


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