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Topics on Kac-Moody Lie algebras

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In this note we introduce Kac-Moody Lie algebras, their representation theory, and relation with theory of differential equations.

1. Kac-Moody Lie algebras ([6], [14], [15])

Let $C = (c_{ij})$ be a $n \times n$-matrix satisfying following conditions;

$$c_{ij} \in \mathbb{Z}, \quad c_{ii} = 2, \quad c_{ij} \leq 0 \quad (i \neq j), \quad c_{ij} = 0 \iff c_{ji} = 0,$$

and define Lie algebra $\mathcal{L}(C)$ over $\mathfrak{c}$ with generators $\{h_i, e_i, f_i \mid i = 1, \ldots, n\}$ and relations:

$$\begin{align*}
[h_i, h_j] &= 0, \\
[h_i, e_j] &= c_{ij} e_j, \\
[h_i, f_j] &= -c_{ij} f_j, \\
[e_i, f_j] &= \delta_{ij} h_i, \\
(\text{ad } e_i)^{c_{ij} + 1} e_j &= 0, \\
(\text{ad } f_i)^{c_{ij} + 1} f_j &= 0.
\end{align*}$$

$\mathcal{L} = \mathcal{L}(C)$ is called Kac-Moody Lie algebra.

$\mathcal{L}$ has grading with $\deg h_i = (0, \ldots, 0)\,$, $\deg e_i = (0, \ldots, 0, 0) = a_i$, and $\deg f_i = -a_i$.

Put $\mathcal{L}_a = \{x \in \mathcal{L} \mid \deg x = a, (a \in \mathbb{N})\}$, $m_a = \dim \mathcal{L}_a$, $\Delta = \{a \in \mathbb{N} \mid m_a \neq 0\}$.

From defining relations $\mathcal{L}$ has vector space decomposition

$$\mathcal{L} = \bigoplus \mathcal{L}_a,$$

where $\mathcal{L}_h = \sum_i \mathcal{L}_h$, $\mathcal{L}_e = \langle e_1, \ldots, e_n \rangle$, $\mathcal{L}_f = \langle f_1, \ldots, f_n \rangle$.

So $\Delta$ is disjoint union of $\Delta_+ = \Gamma_+ \cap \Delta$ and $\Delta_- = (-\Gamma_+) \cap \Delta$.

(\Gamma_+ = \mathbb{Z} a_1 + \cdots + \mathbb{Z} a_n)

For each $i$, define $s_i \in \text{GL}(\mathfrak{n})$ by

$$s_i(a_j) = a_j - c_{ij} a_i,$$

and $W = \langle s_i \mid i = 1, \ldots, n \rangle$. 

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If \( C \) is decomposable, i.e., has a permutation matrix \( P \) such that 
\[
PCP^{-1} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix},
\]
then \( \mathcal{L}(C_1) \cong \mathcal{L}(C_1) + \mathcal{L}(C_2) \), \( \mathbb{W}(C) \cong \mathbb{W}(C_1) \times \mathbb{W}(C_2) \).
So we assume \( C \) is indecomposable.

A finite-dimensional complex simple Lie algebra is isomorphic to a \( \mathcal{L}(C) \)
whose \( \mathbb{W}(C) \) is finite group. Equivalent condition on \( C \) to finiteness of \( \mathbb{W}(C) \)
is that there exist positive numbers \( d_1, \ldots, d_n \) such that 
\[
\begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{pmatrix} C = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}
\]
is a positive definite symmetric matrix.
If we replace "positive definite" by "positive semi-definite" in this
condition, we get a new class of infinite-dimensional Lie algebras. They
are called Euclidean (or affine) Lie algebras. We deal with Euclidean case
hereafter.

2. Character formula and denominator formula

Put \( V = \mathfrak{g} \mathcal{L} \mathcal{R} \mathcal{I} \mathcal{L} \), and for \( \lambda \in \mathfrak{g} \mathcal{L} \mathcal{R} \mathcal{I} \mathcal{L} \mathcal{L} \mathcal{R} \mathcal{L} \mathcal{R} \mathcal{I} \mathcal{L} \mathcal{L} \mathcal{R} \mathcal{I} \mathcal{L} \mathcal{L} \mathcal{R} \mathcal{L} \mathcal{R} \), let \( c_{\lambda} : \mathbb{W} \to V \) be the \( I \)-cocycle, i.e., satisfies 
\[
c_{\lambda}(w_1, w_2) = w_1c_{\lambda}(w_2) + c_{\lambda}(w_1), \text{such that } c_{\lambda}(s_i) = \lambda(a_i) \quad (i = 1, \ldots, n).
\]
If \( \lambda(h_i) \in \mathbb{Z} \) for all \( i \), \( \lambda \) is said integral, and moreover if \( \lambda(h_i) \geq 0 \), \( \lambda \) is
said dominant integral. It is easy to see
\[
\begin{cases}
\lambda : \text{integral} \Rightarrow c_{\lambda}(\mathbb{W}) \subset \mathcal{L} \\
\lambda : \text{dominant integral} \Rightarrow c_{\lambda}(\mathbb{W}) \subset \mathcal{L}_+ \end{cases}
\]
To each dominant integral, there exists \( \mathcal{L} \)-module \( V^\lambda \), unique up to isomorphism,
with the following property;
\[
0 \not\exists v \in V \text{ s.t. } \begin{cases}
e_i v = 0 \\
h_i v = \lambda(h_i) v, \quad f_i \lambda(h_i) v = 0 \quad (i = 1, \ldots, n)
\end{cases}
\]
\[
V^\lambda = \sum \mathbb{C} f_{i_1} f_{i_2} \cdots f_{i_k} v
\]
\( V^\lambda \) is called standard module with highest weight \( \lambda \).
For \( \alpha \in \mathcal{L}_+ \), let
\[
V^\alpha = \sum_{a_1, \ldots, a_n \in \mathbb{Z}} \mathbb{C} f_{i_1} \cdots f_{i_k} v
\]
\( \text{ch} V^\lambda = \sum_{\alpha \in \mathcal{L}_+} \chi_{\mathcal{L}^\alpha}(e(-a_1), \ldots, e(-a_n)) \) (\( \chi \mathcal{L}^\alpha \) is in \( \mathbb{Z} \{[e(-a_1), \ldots, e(-a_n)] \} \)).
Let $\rho \in \kappa$ be defined by $\rho (h_i) = I$ for all $i$. We have

character formula \([5],[8],[10]\)

$$\sum_{w \in \mathbb{Z}} \det(w) e(-c_{A_1 \rho}(w))$$

$$\sum_{w \in \mathbb{Z}} \det(w) e(-c_{A_1 \rho}(w))$$

and
denominator formula

$$\sum_{w \in \mathbb{Z}} \det(w) e(-c(w)) = \prod_{a \in \Delta_+} (1 - e(-a))^{m_a}$$

If we make specialization $e(-a_i) \mapsto q^{m_i}$ ($m_1, \ldots, m_n > 0$) to both sides of

denominator (or character) formula, we have various combinatorial identities.

\([7],[8],[12],[16]\)

3. K-dV equation and construction of representation

Let $u(x), v_{n1}(x)$ ($n=1, 2, 3, \ldots, 1 \leq k \leq 2n-2$) be functions in infinite many

variables $x=(x_1, x_2, x_3, \ldots)$. From the compatibility conditions of linear

partial differential equations

$$\left\{ \begin{array}{l}
\frac{\partial^2}{\partial x_1^2} u(x) + u(x) v(x) = \lambda \psi(x) \\
\frac{2n}{2n+1} u_{n1} + v_{n1} + \cdots + v_{n2n-2} \psi (n=1, 2, \ldots) 
\end{array} \right\}$$

we obtain non-linear differential equations on $u$, and $v_{n1}$ are expressed by
differential polynomials of $u$. These differential equations are called

K-dV equations. A function $\mathcal{Z}(x)$ is called $\mathcal{Z}$-function, if $u = \frac{\partial^2}{\partial x_1^2} \log \mathcal{Z}$ is a

solution of K-dV equations. The space of infinitesimal trasformations of

$\mathcal{Z}$-functions forms a Lie algebra $\mathcal{L}$ of linear trans-formations on $C[x_1, x_3, \ldots]$. Date, Jimbo, Kashiwara and Miwa find \([11]\) that $\mathcal{L}$ is $\mathcal{L}$-module

$C[x_1, x_3, \ldots]$ coincides with the standard module constructed by Lepowsky and

Wilson. \([11]\). Kac, Kazhdan, Lepowsky and Wilson construct standard modules

for other Euclidean Lie algebras in analogous way to $\mathcal{L}$, and D-J-K-M show
they correspond to some non-linear equations like K-dV.([2],[3],[9])

4. Remarks

(1) About classification and realizations of Euclidean Lie algebras, c.f. [6],[15].

(2) Let $\mathfrak{g}$ be the maximam homogenous ideal with $\mathfrak{g} \cap L = 0$. Strictly speaking, $\mathfrak{g}(C) = L(C)/\mathfrak{g}$ is Kac-Moody Lie algebra. But finite or Euclidean case $\mathfrak{g} = 0$, and we conjecture that $\mathfrak{g} = 0$ for any case.

(3) Character formula and denominator formula are analogy of finite case, and proved for $\mathfrak{g}(C)$ with symmetrizable $C$, i.e. there exist positive numbers $d_1, \ldots, d_n$ such that $(d_1 \cdots d_n)C$ is symmetric matrix.

(4) Original K-dV equation is obtained by to put $x_5, x_7, \ldots$ constant.

(5) Frenkel and Kac ([4]) construct standard modules in different way from K-K-L-W. It will be interesting to describe isomorphism explicitly between K-K-L-W's module and F-K's.

References


[7] V.G.Kac, Infinite-dimensional Lie algebras and Dedekind's $\eta$-function,


