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On the Structure of the Set of Gibbs States for the 2-dimensional Ising Ferromagnet

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Let Z^2 be a square lattice, At each point $x \in Z^2$, we put + or - spin. $\Omega \equiv \{+1, -1\}^{Z^2}$ is the set of all possible spin configurations on Z^2. For each finite $V \subset Z^2$ and $w \in \Omega$, the interaction energy in V is given by

$$E_V^w(\sigma) = -\sum_{<x.y> \subset V} \sigma(x)\sigma(y) - \sum_{\substack{x \in V \\ y \in \partial V \\ <x.y>}} \sigma(x)w(y), \quad \forall\sigma \in \{+1, -1\}^V$$

where $\sum_{<x.y> \subset V}$ denotes the summation over all nearest neighbour pairs in V. ∂V denotes the boundary of V, and w is called the boundary condition.

For any $\beta > 0$, a Gibbs state (for the parameter β) of this system is a probility measure satisfying for every finite V, $\sigma \in \{+1, -1\}^V$

$$\mu(\sigma | F_{V^c})(w) = Z(w)^{-1} \exp[-\beta E_V^w(\sigma)] = P_V^w(\sigma)$$

where F_{V^c} is the σ-algebra generated by $\{w(x),\ x \in V^c\}$, and $\mu(\cdot | F_{V^c})(w)$ is the conditional probability of μ on V given the outside configuration w. $Z(w)$ is the normalization.

Let $\mathcal{G}(\beta)$ be the set of all Gibbs states for the parameter $\beta > 0$. Then the following fact is known.

(a) $\mathcal{G}(\beta)$ is convex compact

(b) There is $\beta c > 0$ such that

$$\#\mathcal{G}(\beta) = 1 \quad \beta \le \beta c \quad \text{and} \quad \#\mathcal{G}(\beta) > 1 \quad \beta > \beta c$$

(c) for $\beta > \beta c$, there are two distinct extremal points $\mu+$ and $\mu-$ such that

$$\mu+ = \lim_{V \uparrow Z^2} P_V^+, \qquad \mu- = \lim_{V \uparrow Z^2} P_V^-$$

where $P_V^{\pm}(\sigma)$ corresponds to the boundary condition $w\pm$;

$\omega\pm(x) = \pm 1$ for all $x \in Z^2$

The problem we asked "Are there any other extremal points for (β), $\beta > \beta c$?" This was paused about 10 years ago by Gallavotti and Dobrushin, and was open till last year.

Theorem (Aizenman, Higuchi)

for any $\beta > \beta c$, we have

$$\mathcal{G}(\mu) = \{\lambda\mu + + (1-\lambda)\mu - \ ; \ \lambda \in [0, 1]\}$$

Remark: In the 3-dimensional case, the above theorem doesn't hold. Dobrushin has shown an example for sufficiently large $\beta > 0$, which cannot be a convex combination of $\mu+$ and $\mu-$.

References.

[1] Aizenman, M ; Translation invariance and instability of phase coexistence in the two dimensional Ising system. Comm. Math. Phys. 73, 83-94 (1980).

[2] Higuchi, Y. ; On the absence of non-translationally invariant Gibbs states for the two-dimensional Ising model. Proc. Conf. on Random Fields, Esztergom. (to appear from North Holland).