数理解析研究所講究録
439巻 1981年 6-17

Random Walks and Some Problems concerning Lorentz Gas

Ya. G. Sinai (Landau Institute)

§1. Random Walks with Random Transition Probabilities

Random walks with random transition probabilities arise in many problems of probability theory and mathematical physics. In particular the corresponding problems have many common features with problems related to the Schrödinger equation with random potentials. The one-dimensional random walks with random probabilities were studied intensively in a series of papers by H. Kesten, M. Kozlow, F. Spitzer and others (see, for example, [1], [2] and [3]).

Let us recall some definitions. We consider the random walk on the usual cubic d-dimensional lattice $\mathbb{Z}^d$. The unit vectors directed along positive coordinate semi-axes are denoted by e_α, $\alpha=1,\cdots,d$. The transition probabilities for the considered random walk have the form

$$Pxx' = \begin{cases} \frac{1}{2d} + \xi_{x,\alpha}, & \text{for } x'=x+e_\alpha,\ \alpha=1,\cdots,d, \\ \frac{1}{2d} - \xi_{x,\alpha}, & \text{for } x'=x-e_\alpha,\ \alpha=1,\cdots,d, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\xi_{x,\alpha}$ are identically distributed symmetric independent random variables with $|\xi_{x,\alpha}| \leqslant \varepsilon_0 < 1 - 1/2d$. We shall denote all probabilities and expectations related to the random variables $\xi_{x,\alpha}$ by P_ξ and E_ξ respectively and by $X(n)$ the position

of the moving point at the moment of time equal to n. From the symmetry of distribution it follows that $E_\xi \xi_{x,}^{2k+1} = 0$ for all $k \geqq 0$. We shall call the random walks with such transition probabilities as random walks with random transition probabilities. In the papers [1] - [3], the authors used the term "random walks in random environments." Certainly one can easily imagine natural generalizations of the introduced concept. The results of the papers [1] - [3] concerned the one-dimensional case and mainly the properties of recurrence of the random walks.

In our paper [4] the limit behaviour of the random walk was considered in the one-dimensional case. Assume that $\alpha>0$ and $\delta>0$ are given. Then the main result of [4] is the following theorem.

<u>Theorem.</u> For all sufficiently large n, one can find a subset C_n in the space of all possible realizations $\xi = \{\xi_{x,\alpha}\}$ and a functional $m^{(n)}(\xi)$ defined on C_n such that

1) $P_\xi(C_n) \geqq 1 - \alpha$;

2) if $\xi \in C_n$ then $P(|x(n)/\ell n^2 n - m^{(n)}| < \delta) \to 1$ as $n \to \infty$ uniformly in $\xi \in C_n$, here P means the probability distribution in the space of random trajectories $X(n)$ as all $\xi_{x,\alpha} \in C_n$ are chosen;

3) the functional $m^{(n)}(\xi)$ has a limit distribution as $n \to \infty$.

This theorem shows that the properties of one-dimensional random walks with random probabilities differ significantly from the properties of the usual random walks. In particular $x(n)$ takes values of order $\ell n^2 n$ for large n. Moreover $x(n)/\ell n^2 n$ turns out to be localized. Apparently this property is of the same nature as the property of the one-dimensional Schrödinger equa-

tion with random potential to have pure-point spectrum with probability 1 (see [5]).

Concerning the multi-dimensional case, we shall mention a result by D. Szasz et al. [6] concerning the local perturbations of the usual random walks. Namely they considered the case when $\xi_{x,\alpha}$ are equal to zero everywhere except a finite set. For this case they have shown under some natural assumptions that $x(n)/\sqrt{n}$ has in the limit $n \to \infty$ Gaussian probability distribution.

§2. Random Walks with Inner Degrees of Freedom

We shall describe in this section another generalization of random walks. Assume that a finite or countable set I is given. We shall call I the space of inner states of the moving point. Suppose now that the moving point is at any moment of time in a point $x \in \mathbb{Z}^d$ and in a state $i \in I$. Thus the state of moving point $X(n)$ is a pair $X(n) = (x(n), i(n))$. We shall consider homogeneous random walks which are defined by the set of transition probabilities $P_y(i,j)$ where $y = \pm e_\alpha$, $\alpha = 1,\cdots,d$ and $i,j \in I$. Namely $P_y(i,j)$ is the conditional probability for the transition $(x(n),i) \to (x(n) + y, j)$. Therefore $P_y(i,j) \geqq 0$, $\sum_{y,j} P_y(i,j) = 1$. We shall call the random walks defined by the set $\|P_y(i,j)\|$ as homogeneous random walks with inner degrees of freedom. The random walk is called symmetric if $P_y(i,j) = P_{-y}(j,i)$.

<u>Definition 1.</u> A measure λ on the space I is called a stationary measure for the random walk if the measure μ_0 on the space $\mathbb{Z}^d \times I$ of pairs (x,i) defined by the formula $\mu_0(x,i)=\lambda_i$

is the stationary measure for the considered random walk with inner degrees of freedom, i.e. $\lambda_j = \sum_{i,y} \lambda_i P_y(i,j)$, $j \in I$.

In the case of symmetric random walks, we can rewrite the last equation in the following form

$$\sum_{i,y} \lambda_i P_y(i,j) = \sum_{i,y} P_{-y}(j,i)\lambda_i = \lambda_j .$$

Let $q(j,i) = \sum_y P_y(j,i)$. Then we have $\sum_i q(j,i)\lambda_i = \lambda_j$. The matrix $Q = ||q(j,i)||$ is a stochastic matrix and we can put $\lambda_j = 1/\#(I)$. Under some natural conditions, this vector is unique.

The theory of symmetric random walks with inner degrees of freedom can be sufficiently far developed. For example, we shall prove the following theorem.

<u>Theorem.</u> Let a symmetric random walk with inner degrees of freedom be given. Assume that I is finite and for some $m > 0$ the matrix Q^m has strictly positive matrix elements. Then for $d = 1,2$ the symmetric random walk is recurrent while for $d \geq 3$ it is transient.

<u>Proof.</u> We shall give only an outline of the proof. Let us denote $\#(I) = m$ and consider the space $L_I^2(\mathbb{Z}^d)$ of vector-valued function $\phi = \{\phi(x), x \in \mathbb{Z}^d\}$ whose values are vectors of $\mathbb{C}^m$ with the usual norm. We introduce an operator A

$$(A\phi)(x) = \sum_{x':x'+y=x} P_y \phi(x').$$

In the case of symmetric random walks this operator is self-adjoint. After Fourier-transform, it becomes an operator-valued function $\hat{A}(\lambda)$ acting on the space $L^2_I(\mathrm{Tor}^d)$ of square-integrable functions $\hat{\phi} = (\hat{\phi}(\lambda),\ \lambda \in \mathrm{Tor}^d)$, $\hat{\phi}(\lambda) \in \mathbb{C}^m$, i.e.

$$(A\hat{\phi})(\lambda) = \hat{A}(\lambda)\hat{\phi}(\lambda).$$

$\hat{A}(\lambda)$ for every $\lambda \in \mathrm{Tor}^d$ is a self-adjoint operator, $\|\hat{A}(\lambda)\| < 1$ everywhere except $\lambda = 0$. At the point $\lambda = 0$ the operator has an eigen-value 1, while all other eigenvalues have absolute values less than 1. Let us denote the corresponding eigen-vector by $e(\lambda)$, i.e. $\hat{A}(\lambda)e(\lambda) = a(\lambda)e(\lambda)$, $a(0) = 1$. It is correctly defined in a sufficiently small neighbourhood of $\lambda = 0$.

Now we introduce the conditional probabilities $q^{(n)}_{x,z}(i,j)$ which are the probabilities of random trajectories which start at (x,i) and after n steps come to the state (z,j). We have the system of recurrent equations

$$Q^{(n+1)}_{x,z} = \sum_y P_y Q^{(n)}_{x+y,z}$$

where $Q^{(n)}_{x,z}$ is the matrix $\|q^{(n)}_{x,z}(i,j)\|$. This matrix depends only on the difference $z - x$. The probabilities $q^{(n)}_{x,x}(i,j)$ can be written after Fourier transform in the form

$$q^{(n)}_{x,x}(i,j) = \int_{\mathrm{Tor}^d} (A^n(\lambda)\delta_i(\lambda),\ \delta_j(\lambda))d\lambda,$$

where $\delta_i(\lambda)$ and $\delta_j(\lambda)$ are suitable $\mathbb{C}^m$-valued functions on Tor^d. We can rewrite it as follows:

$$q^{(n)}_{x,x}(i,j) = \int a^n(\lambda)\ \delta_i^{(0)}(\lambda)\cdot\delta_j^{(0)}(\lambda)d\lambda + r^{(n)}_{x,x}\ (i,j).$$

Here $\delta_i^{(0)}(\lambda)$ and $\delta_j^{(0)}(\lambda)$ are projections of $\delta_i(\lambda)$ and $\delta_j(\lambda)$ onto the subspace generated by $e(\lambda)$ and the integration goes over the neighbourhood of $\lambda = 0$, where $e(\lambda)$ is correctly defined, $r^{(n)}_{x,x}(i,j)$ is a remainder term. It is easy to see that $r^{(n)}_{x,x}(i,j)$ decays exponentially fast while the first term has the same asymptotic behaviours as the transition probability after n steps in the case of the usual random walk. After this the arguments concerning the recurrence properties can be carried out and we get the desired result.

§3. Lorentz Gas and Random Walks with Inner Degrees of Freedom.

Let $S = \{s_i\}$ be a countable subset of $\mathbb{R}^2$ such that $\| s_i - s_j \| > 2\rho$ for some $\rho > 0$, $i \neq j$. We denote by $D_\rho(s_i)$ the disk of the radius centered at s_i and $D = \bigcup_i D_\rho(s_i)$. These disks are called scatterers and D is a configuration of scatterers. Lorentz Gas is the dynamical system generated by the motion of a single particle between the disks with constant velocity and with elastic reflections from the boundaries $\partial D_\rho(s_i)$ of disks $D_\rho(s_i)$. Lorentz proposed this model more than seventy years ago for the description of the motion of electrons in metals. He assumed that the scatterers are heavy ions which suppose to be unmoving while electrons are light particles and

we can neglect their inner interaction. Since that time the Lorentz Gas is one of popular models of non-equilibrium statistical mechanics.

We shall assume that the velocity of the particle is equal to 1. Let C_i be the set of unit vectors beginning at a point of $\partial D_\rho(s_i)$ and directed outside the boundary. Topologically C_i is a cylinder and we can introduce natural coordinates (r,ϕ) on it, where r is the cyclic coordinate along the boundary $\partial D_\rho(s_i)$, $0 \leq r < 2\pi\rho$, and ϕ is the angle between the vector beginning at the point with the coordinate r and the unit normal vector $n(r)$ beginning at r and directed outside $\partial D_\rho(s_i)$, $-\pi/2 \leq \phi \leq \pi/2$. We consider the set $c = \bigcup_i C_i$ and the measure μ on C whose restriction to C_i has the form $d\mu = \cos\phi\, dr\, d\phi$. It is obvious that C is the measure space with the σ-finite measure μ.

Suppose that for $x \in C_i$ the straight segment which begins at x intersects a scatterer $D_\rho(s_j)$. We can construct $y \in C_j$ assuming that we have a part of a trajectory of the Lorentz gas between two subsequent reflections (see Figure 1). In what follows we shall

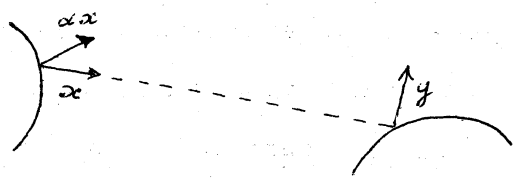

Figure 1.

consider only such S for which for μ-almost all x one can find the corresponding y. Then we get a transofrmation T of the

measure space (C,μ) where $Tx = y$. It is an analogy of the usual Liouville theorem that T preserves the measure μ. On the space C one can introduce the natural involution α which transforms $x = (r,\phi)$ into $\alpha x = (r,-\phi)$ (see Figure 1).

Suppose now that D is a periodic configuration of scatterers. It means that one can find a discrete subgroup Γ of translations of the plane and a parallelogram Π_0 such that if $\Pi_\gamma = \gamma\Pi_0$ then $(\Pi_{\gamma_1} \cap \Pi_{\gamma_2}) = \partial\Pi_{\gamma_1} \cap \partial\Pi_{\gamma_2}$, $\bigcup_{\gamma\in\Gamma} \Pi_\gamma = \mathbb{R}^2$. Periodicity of D means that $\Pi_\gamma \cap D = \gamma(\Pi_0 \cap D)$. We shall assume also that D is such that the length of the segment connecting x and y is uniformly bounded.

Suppose that we have a finite or countable partition ξ of the set C which is also Γ-periodic, i.e. if A_ξ is an element of ξ then γA_ξ is also an element of ξ. The partition ξ is called symmetric if each A_ξ is contained in a C_i and αA_ξ is also an element of the partition ξ. We denote $C_0 = \bigcup_{i:s_i\in\Pi_0} C_i$ and $\xi_0 = \xi|C_0$. Let the elements of ξ_0 be $(A_1, A_2,\cdots, A_n,\cdots)$ and I be the set of indices $(1,2,\cdots,n,\cdots)$. We shall say that the point $x \in C_k$ is in the state (γ,j) if $s_k \in \Pi_\gamma$ and $x \in \gamma A_j$.

For every semi-trajectory $x_0, Tx_0,\cdots, T^m x_0,\cdots$, we can construct the sequence of pairs $(\gamma_0,j_0), (\gamma_1,j_1),\cdots, (\gamma_m,j_m),\cdots$, in such a way that $T^m x_0$ is in the state (γ_m,j_m).

Γ can be considered as the usual lattice of points (p,q) where p and q are integers. We introduce the norm $\|\gamma\| = |m| + |n|$. One can choose Γ so large that $\|\gamma_m - \gamma_{m-1}\| \leq 1$.

Thus we can consider the sequence of points $(\gamma_0,j_0),(\gamma_1,j_1),\cdots,(\gamma_m,j_m),\ldots,$ as a trajectory of the random walk with inner degrees of freedom. The measure μ induces a natural measure on the space of all trajectories which is translation-invariant.

It is a natural problem to choose ξ in such a way that the induced measure could be sufficiently well approximated by Markov measures.

The construction of such ξ is based upon the notion of a Markov partition. Firstly we recall the notion of a parallelogram. Let $x \in C$. A smooth curve γ is called local stable (unstable) manifold of the point $x \in \gamma$ if T^n is continuous on γ for all $n \geqq 0$ ($n \leqq 0$) and diam $(T^n\gamma) \to 0$ as $n \to \infty$ ($n \to -\infty$). It is known that for μ-almost all x local stable (unstable) manifolds exist (see e.g. [7]). In what follows we shall mean under $\gamma^{(s)}(x)$ and $\gamma^{(u)}(x)$ the maximal local stable and unstable manifolds respectively.

A subset $P \subset C_i$ is called a parallelogram if for every $x, y \in P$ the intersection $\gamma^{(s)}(x) \cap \gamma^{(u)}(y)$ P (see [8]). Each parallelogram P has natural partitions $\xi_P^{(s)}$ and $\xi_P^{(u)}$ whose elements are of the form $\gamma^{(s)}(x) \cap P$ and $\gamma^{(u)}(x) \cap P$ respectively for $x \in P$. We shall denote by $C_{\xi_P^{(s)}}(x)$ $(C_{\xi_P^{(u)}}(x))$ the intersection $\gamma^{(s)}(x) \cap P$ $(\gamma^{(u)}(x) \cap P$ respectively).

<u>Definition 2.</u> A partition ξ of the measure space (C,μ) is called a Markov partition if its elements are parallelograms $P_1, P_2, \cdots,$ and for almost every $x \in P_i \subset C$, $Tx \in P_j$, $T^{-1}x \in P_k$,

we has the inclusions

$$TC_{\xi_{P_i}^{(s)}}(x) \subset C_{\xi_{P_j}^{(s)}}(Tx), \quad T^{-1}C_{\xi_{P_i}^{(u)}}(x) \subset C_{\xi_{P_k}^{(u)}}(T^{-1}x).$$

In the paper [9] L.A.Bunimovich and I have constructed a periodic Markov partition for a periodic configuration of scatterers, symmetric under the involution α .

Using the Markov patitions we constructed natural Markov approximations for the measure μ . Some consequences follow from the properties of these approximations. We shall mention one of them (see [10] for this and other results).

Let $q(t)$, $0 < t < \infty$, be a trajectory of Lorentz gas on the configuration space $\mathbb{R}^2 \setminus D$. We suppose that the initial point $x(0) = (q(0), v(0))$ is random and is distributed according to a probability distribution with smooth density. For every $T > 0$ we consider continuous functions $[0,1] \to \mathbb{R}^2$ defined by the formula

$$q_T(s) = \frac{1}{\sqrt{T}} q(sT), \quad 0 \leq s \leq 1.$$

The probability distribution for $x(0)$ induces the probability measure P_T on the space of functions $q_T(s)$.

<u>Theorem</u>. P_T converges weakly to the Wiener measure as $T \to \infty$.

Markov partitions give the possibility to reduce problems related to Lorentz gas to problems concerning random walks.

The construction of Markov partitions can be carried out in many cases. Let us give some examples.

1. Suppose that $D^{(0)}$ is a periodic configuration of scatterers. We shall mean under local perturbation of $D^{(0)}$ any configuration D which differs form $D^{(0)}$ by a position of a finite set of scatterers. Markov partition in this case is also a local pertubation of Markov partitions for periodic configurations $D^{(0)}$ and the problems can be formulated in terms of random walks with inner degrees of freedom with local perturbation of transition probabilities. We come to a problem of a generalization of results by D. Szasz ([6]) mentioned above to the case corresponding to the Lorentz gas. Some results in this problem are already proven but the general problem still remains open.

2. Suppose again that $D^{(0)}$ is a periodic configuration of the scatterers and let us shift each scatterer independently from the other scatterers. In this way we get a purely random configuration of scatterers D. Markov partitions can be constructed in this case too. Because D is random, Markov partitions are also random and we come to a generalization of random walks with inner degrees of freedom with random transition probabilities.

Probably many dynamical systems with large number degrees of freedom can be represented as random walks with random transition probabilities.

References

[1] H. Kesten, M.V. Kozlov and F.Spitzer: A limit law for random walk in a random environment, Comp. Math., 30(1975), 145-168.

[2] G. Solomon: Random walks in a random environment, Ann. Prob., 3(1975), 1-31.

[3] H. Kesten: A renewal theorem for random walk in a random environment, Proc. Symp. Pure Math., 31(1977), 67-77.

[4] Ya. G. Sinai: Theory Prob. Appl., (in press).

[5] I. Ya. Goldsheid, S. A. Molchanov and L. A. Pastur: A pure point spectrum of the stochastic one-dimensional Schrödinger operator, Func. Analy. Appl., 11(1977), 1-8.

[6] D. Szasz: Private Communication.

[7] Ya. G. Sinai: Dynamical systems with elastic reflections, Russian Math. Surveys, 25(1970), 137-189.

[8] R. Bowen: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture Notes in Math. 470, Springer-Verlag, 1975.

[9] L. A. Bunimovich and Ya. G. Sinai: Markov partitions for dispersed billiards, Comm. Math. Phys., 78(1980), 247-280.

[10] _____: Statistical properties of Lorentz gas with periodic configuration of scatterers, ibid., 78(1981), 479-497.