数理解析研究所講究録
441巻 1981年 66-82

Definability problems in metric spaces; a summary

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Introduction

We have been investigating the "definability problem" in analysis. In practice it is a program to develop mathematical theories in formal systems which are modest extensions of Peano arithmetic, thus establishing the soundness of the theories relative to the given mathematical structures. The formal systems we have employed in our forgoing researches are those based on many sorted logic in which the arithmetically definable theory of reals can be developed and which have "definable" inductive definitions of ω-type. For the detailed discussion of our standpoint, see [5] and [6].

A recent result along this line concerns the elementary theory of topology. Here as a sequel to it we work on metric spaces. Since much has been discussed of proof-theoretical content of our endeavor, we shall place more emphasis this time on mathematical aspects of the theme, thereby explaining by examples how to construct various mathematical objects within our language.

Let us remark here the following. The notion of definability, which is to be defined, is preserved under set theoretical operations such as intersection, complement, closure, interior.

(See [6] for detail.)

§1 Symbols and axioms

Definition 1.1. Atomic types are three sorted; one for rationals and two for the elements of two spaces. (For the sake of simplicity, we assume two spaces.) Compound types (of predicate type) are defined as usual.

Definition 1.2. The language consists of the symbols of the definable theory of reals, FA (see [3]), and the following:

X, Y, $eq(X; \, , \,)$, $eq(Y; \, , \,)$, ρ, σ, x_0, y_0, I_0, I_1, I_2, ...;

variables for each type.

The intended interpretations and types of those symbols should be clear; I_i denotes the i-th constant for the inductive definition.

Definition 1.3. Definability, terms, formulas, abstracts and sequents are defined as were in [6]. Let us pick up a few crucial cases.

1) An object is said to be definable if the only quantifiers it may contain are of atomic type.

2) An expression of the form $\{\psi_1,\dots,\psi_n\}F(\psi_1,\dots,\psi_n)$, where $F(\psi_1,\dots,\psi_n)$ is a definable formula, is an abstract.

3) If Φ is an I_i or a free variable and $J_1,\dots,J_n$ are terms or abstracts of appropriate type, then $\Phi(J_1,\dots,J_n)$ is a formula.

Definition 1.4. Substitution is defined as in [3].

Definition 1.5. Logical system L. The logical system L is the predicate calculus of our language augmented by the comprehension rules applied to our definable abstracts.

Definition 1.6. There are three sets of axioms.

1) A: the set of axioms of arithmetic, where the mathematical induction and the equality axiom are formulated in terms of the higher universal quantifiers. (See [3]. We follow the notational convention there.)

2) B: the set of axioms on (X,ρ) and (Y,σ) as metric spaces. For better understanding, we shall employ conventional mathematical notations. In particular, eq$(X;x,y)$ will be denoted by $x = y$ and $\{t\}\rho(x,y,t)$ will be abbreviated to $\rho(x,y)$.

$\forall x(x\varepsilon X)$; $x_0\varepsilon X$;

$\forall x\forall y R(\rho(x,y))$;

the equivalence relation on $=$;

the metric property of ρ.

With (Y,σ) likewise.

3) C: the set of axioms on ω-type definable inductive definition, which has been presented in [6].

Definition 1.7. A sequent $\Gamma \to \Delta$ of our language is said to be a theorem of M_0 if

$$A,\ B,\ C,\ \Gamma \to \Delta$$

is provable in the system L. M_0 will be called a theory of metric spaces.

§2 Topology

Remark. Here and in all the sections that follow, the propositions are meant to be the theorems of M_0. Also, we assume the definable theory of reals throughout (cf. [3]).

Definition 2.1. Q will denote the set of positive rationals, while r, s, t, ε, δ will stand for rationals. $\Lambda \equiv (X,Q)$. We write λ for an element of Λ. $S(x,r)$ abbreviates $\{y\}(\rho(x,y)<r)$.

Proposition 2.1.
$$r_1 > 0 \wedge r_2 > 0 \wedge y\varepsilon S(x_1,r_1) \cap S(x_2,r_2)$$
$$\rightarrow \exists r > 0\ (r<\min(r_1-\rho(y,x_1),r_2-\rho(y,x_2))$$
$$\wedge\ y\varepsilon S(y,r) \subset S(x_1,r_1) \cap S(x_2,r_2)).$$

This leads us to the following.

Proposition 2.2. $\{S(x,r);\ x\varepsilon X, r\varepsilon Q\}$ satisfies the axioms of the base of topology with the index set (X,Q).

Theorem. The elementary theory of the topology induced by ρ in Proposition 2.2 is sound relative to definable instantiations of the given metric space.

Proof. Proposition 2.2 is established in M_0, hence is sound relative to definable instantiations of the given space. On the other hand, it has been established in [6] that the elementary theory of topology is sound relative to definable instantiations of a given base as well as other axioms.

We do not repeat the definitions of various objects; the reader should refer to [6]. Let us show just one example.

opn(A): $ss(X,A) \wedge \forall x \varepsilon A \exists r > 0(S(x,r) \subset A)$

(A is an open subset of X.)

§3 Separability, countability and normality

Proposition 3.1. $\forall x \forall r > 0\ \exists ! y(\phi(x,r,y) \wedge \rho(x,y) < r)$

$\rightarrow$ "The first countability axiom holds with regards to ϕ"

by a definable S^*.

Proof. Define

$S^*(x,n)$: $\{z\}(\phi(x,1/n,y) \wedge \rho(x,y) < 1/n \wedge z \varepsilon S(y,1/n))$.

Then $\{n\}S^*(x,n)$ forms a countable base for x.

Definition 3.1. $MS(X,E)$: $\forall n \exists y \forall z(E(n,z) \leftrightarrow z=y)$

$\wedge \forall x \forall r>0 \exists n \exists y(E(n,y) \wedge \rho(y,x)<r)$,

where E is a parameter. (X is metric separable by E.) We write e_n for the y satisfying $E(n,y)$.

Proposition 3.2. (Equivalence of metric separability and second countability)

1) $MS(X,E)$ "$\{n,m\}S(e_n,1/m)$ forms a countable base for the topology defined in Proposition 2.2 and $e_n\ \varepsilon \{n,m\}S(e_n,1/m)$."

2) $opnsq(\Phi) \wedge sq(\phi) \wedge \forall n \forall x(\phi(n,x) \vdash x \varepsilon \Phi(n))$

$\wedge \forall x \forall r > 0(S(x,r) = \bigcup \{\Phi(i)\ ;\ \Phi(i) \subset S(x,r)\}) \rightarrow MS(X,\phi)$.

Definition 3.2. $\rho(x,F) = \inf\{\rho(x,y); y \varepsilon F\}$, which is definable. $\rho(x,F)$ is a real.

Proposition 3.3. A metric space is normal.

Proof. Define ρ_1 and ρ_2 as follows.

$$\rho_1(F,G) = \{x\}(\rho(x,F) < \rho(x,G)),$$
$$\rho_2(F,G) = \{x\}(\rho(x,G) < \rho(x,F)).$$

X is normal by ρ_1 and ρ_2.

§4 Sequences and convergence

Sequence, convergence, cluster point, subsequence, etc. are defined as in [6]. Those notions can be equivalently reformulated in terms of the metric.

§5 Continuous functions

Here we consider two metric spaces (X,ρ) and (Y,σ). We assume $x, u, z, \ldots \varepsilon X$ and $y, v, \ldots \varepsilon Y$; $\lambda \varepsilon \Lambda \equiv (X,Q)$ and $\mu \varepsilon M \equiv (Y,Q)$.

Proposition 5.1. $\mathrm{cnt}(f,X,Y) \leftrightarrow \forall \varepsilon > 0 \forall x \exists \delta > 0 \forall u(\rho(x,u) < \delta \vdash \sigma(f(x),f(u)) < \varepsilon)$,

where

$\mathrm{cnt}(f,X,Y)$: $\mathrm{mp}(f,X,Y) \wedge \forall \mu\ \mathrm{opn}(\mathrm{inv}(f,S(\mu)))$.

Proof. $(\rightarrow)$ Given $\varepsilon > 0$ and $x \varepsilon X$. Put $O \equiv \{y;\ \sigma(f(x),y) < \varepsilon\}$; O is open in Y and inv(f,O) is open in X. So,

$$\exists z \exists r > 0(x \varepsilon S(z,r) \subset \mathrm{inv}(f,O)).$$

Put $a = \rho(z,x)$. a is a positive real and $\exists \delta > 0\ (\delta < r-a)$. If $\rho(x,u) < \delta$, then

$$\rho(z,u) \leq \rho(z,x) + \rho(x,u) < a + r - a = r.$$

So $u \varepsilon S(z,r) \subset inv(f,O)$, hence $f(u) \varepsilon O$, or $\sigma(f(x),f(u)) < \varepsilon$.

The proof of the converse is omitted.

Definition 5.1. $unfcnt(f)$: $\forall \varepsilon > 0 \exists \delta > 0 \forall x \forall u(\rho(x,u) < \delta \vdash \sigma(f(x),f(u)) < \varepsilon)$.

Proposition 5.2. $unfcnt(f) \wedge Csq(X,S) \rightarrow Csq(Y,f(S))$,
where $Csq(X,S)$ expresses that S is a Cauchy sequence from X.

Various properties concerning homeomorphism can be stated and proved as in the general setting.

Definition 5.2. $cmpl(X,\rho)$: $\forall S(Csq(S) \vdash \exists x cnv(S,x))$

Proposition 5.3. Define $g^*(A,f,\zeta,x,y)$ to be $cnv(f(\zeta(x)),y)$. Then:

$cmpl(Y,\sigma) \wedge A \subset X \wedge unfcnt(f,A,Y)$
$\wedge \forall x \varepsilon cl(A)(sq(A,\zeta(x)) \wedge cnv(\zeta(x),x))$
$\rightarrow unfcnt(g^*(A,f,\zeta),cl(A),Y)$
$\wedge g^*(A,f,\zeta)\restriction A \equiv f \wedge$ "such a g^* is unique on $cl(A)$",
where ζ serves as a relation of a sequence associated with $x \varepsilon cl(A)$.

§6 Subspaces

Proposition 6.1. $ss(X,C) \rightarrow$ "$\langle C,\rho\restriction C\rangle$ is a metric space."

Proposition 6.2. Let κ be an enumeration of (N,N). There is a definable E^* such that

$$ss(X,C) \wedge MS(X,E) \wedge sq(C,\phi)$$
$$\wedge \forall i(i = \kappa(n,m) \wedge C \cap S(e_n,1/m) \neq \phi$$
$$\vdash \phi_i \varepsilon C \cap S(e_n,1/m)) \rightarrow MS(C,E^*).$$

Proof. Let P(i) be

$$i = \kappa(n,m) \vdash \phi_i \varepsilon C \cap S(e_n,1/m).$$

Define $\nu^*(i)$ by:

$$\nu^*(0) = \min(i,P(i)),$$
$$\nu^*(j+1) = \min(i,i>\nu^*(j) \wedge P(i)).$$

Put $O_j \equiv C \cap S(e_n,1/m)$ where $\nu^*(j) = \kappa(n,m)$. Now define

$$E^*(j,y): \quad y = \phi(\nu^*(y)).$$

Other properties concerning subspaces are stated and proved as usual.

§7 Baire category theorem

Definition 7.1. $dns(A)$: $ss(X,A) \wedge \forall\lambda\exists x \varepsilon A \cap S(\lambda)$

$nwd(A)$: $ss(X,A) \wedge cl(X-cl(A)) = X$

Proposition 7.1. $nwd(A) \leftrightarrow \forall x \forall r(S(x,r) \subset cl(A))$.

Definition 7.2. $FC(A,\Phi)$: $\forall i\, nwd(\Phi(i)) \wedge A = \smile\Phi$.

(A is of first category by Φ.)

Proposition 7.2. Let Ψ, E, R be parameters.

$$\mathrm{cmpl}(X) \wedge [\forall i(\mathrm{opn}(\Psi(i)) \wedge \mathrm{dns}(\Psi(i))$$
$$\wedge \forall e_0 \forall r_0 > 0\{\mathrm{sq}(E) \wedge E(0) = e_0$$
$$\wedge \text{"R is a sequence of positive rationals"}$$
$$\wedge R(0) = r_0 \wedge \forall n(R(n) \leq n^{-1})$$
$$\wedge \forall n(S(e_0,r_0) \cap \Psi(n) \neq \phi$$
$$\wedge \mathrm{cl}(S(e_n,r_n)) \subset S(e_{n-1},r_{n-1}) \cap \Psi(n))\}]$$
$$\rightarrow \text{"}\bigcap\{\Psi(n);\ n=1,2,\ldots\}\text{ is dense and open."}$$

Note. If we write e_n for E(n) and r_n for R(n), and if { } is satisfied, then we say that $\{e_n\}$ and $\{r_n\}$ are associated with (e_0,r_0) and Ψ.

Proposition 7.3. (Baire category theorem). Assume cmpl(X). For any $\{Y_n\}$ a sequence of nowhere dense sets in X, where $\{X\text{-cl}(Y_n)\}$ and (e_0,r_0) have associated sequences as in Proposition 7.2, (e_0,r_0) being arbitrary, X is not the union of $\{Y_n\}$.

Proposition 7.4. (The uniform boundedness principle). Assume cmpl(X). Let F and M be parameters. Suppose F is a sequence of real-valued continuous functins and M is a map from X to reals satisfying

$$\forall x \forall n\ |F(n,x)| \leq M(x).$$

Define

$E(n,m)$: $\{x;\ |F(n,x)| \leq m\}$,

E_m: $\bigcap\{E(n,m);\ n=1,2,\ldots\}$.

Suppose (*) below holds.

(*) $\{X\text{-}E_m\}$ and (e_0,r_0) have associated sequences E and R as in Proposition 7.2 for any (e_0,r_0).

Then

$$\exists m_0 \exists e \exists r > 0 \forall n \forall x \varepsilon S(e,r)(|F(n,x)| \leq m_0).$$

Proof. By Propositions 7.1 - 7.3.

§8 Compactness

Definition 8.1. $MC(X,E,\Phi)$: $MS(X,E)$

$\wedge$ "X is sequentially compact by Φ"

(X is metric compact by E and Φ)

See [6] for sequential compactness.

Proposition 8.1. Elementary theorems of topology concerning sequential compactness are valid here.

Proposition 8.2. There are definable N^* and σ^* such that

$$MC(X,E,\Phi)$$
$$\rightarrow \forall \varepsilon > 0 \forall x \varepsilon X \exists k \leq N^*(\varepsilon)(\rho(x,E(\sigma^*(\varepsilon,k))) < \varepsilon).$$

Proof. Define

$$\sigma^*(\varepsilon,1) = 1,$$
$$\sigma^*(\varepsilon,n+1) = \min(m, \forall i \leq n \rho(E(\sigma^*(\varepsilon,i)),E(m)) \geq \varepsilon/2),$$

and put

$$N^*(\varepsilon) = \min(m,\ 1 < m \wedge \sigma^*(m) = 1).$$

Definition 8.2. $TB(X,\psi,\nu)$: $\forall \varepsilon > 0\{\forall n \leq \nu(\varepsilon) \exists ! y \psi(\varepsilon,n,y)$

$\wedge\ \forall x \varepsilon X \exists k \leq \nu(\varepsilon)\psi(\varepsilon,k,y) \wedge \rho(x,y) < \varepsilon\}$

(X is totally bounded by ψ and ν.)

Proposition 8.3. $MC(X,E,\Phi) \rightarrow TB(X,\psi^*,N^*) \wedge cmpl(X)$, where N^* was defined in Proposition 8.2 and $\psi^*(\varepsilon,n,y)$ is defined to be $y = E(\sigma^*(\varepsilon,n))$.

Proposition 8.4. $cmpl(X) \wedge TB(X,\psi,\nu) \rightarrow MC(X,\Phi^*,E^*)$ for definable Φ^* and E^*.

Proof. Define

$$E^* \equiv \bigcup\{\psi(1/m,n);\ m=1,2,\ldots,n\leq\nu(1/m)\},$$

where $\psi(\varepsilon,n)$ represents the y such that $\psi(\varepsilon,n,y)$. The construction of Φ^* can be explained as follows. First define

$$m_1=\min(m,m\leq\nu(1)\wedge\forall n\exists\ell\geq n(x_\ell\,\varepsilon\,S(\psi(1,m),1))),$$

$$m_{k+1}=\min(m,m\leq\nu(1/(k+1))$$
$$\wedge\forall n\exists\ell\geq n(x_\ell\,\varepsilon\bigcap\{S(\psi(1/i,m_i),1/i);i\leq k\}\cap S(\psi(1/(k+1),m),1/(k+1)))),$$

where we write $\{x_n\}$ for a sequence. Then define

$$\ell_1=\min(i,x_i\,\varepsilon\,S(\psi(1,m_1),1)),$$

$$\ell_{k+1}=\min(i,\ell_k<i\wedge x_i\,\varepsilon\bigcap\{S(\psi(1/j,m_j),1/j);j\leq k+1\}).$$

Now $\Phi^*(\{x_n\})$ is defined to be $\{k\}\{x_{\ell_\kappa}\}$.

Note. Since metric compactness and total boundedness together with completeness are definably interpretable of one another, we shall use either of them as the definable notion of compactness.

Proposition 8.5. A closed subset Y of a metric compact space is metric compact, presuming that the (*) below holds.

(*) $\forall\varepsilon>0\{\forall i\exists y\varepsilon Y\rho(E(\varepsilon/2,i),y)<\varepsilon/2\vdash\exists!y(\psi(\varepsilon/2,i,y)\wedge\rho(E(\varepsilon/2,i),y)<\varepsilon/2)\}$,

where ψ is a parameter.

Proof. Y is separable by $\{\varepsilon,i\}\psi(\varepsilon/2,i)$.

Proposition 8.6. 1) If unfcnt(f,X,R) and X is totally bounded, then sup f and inf f exist.

2) A metric compact subset Y of a metric space is closed.

§9 Product space

Definition 9.1. We consider a sequence of metric spaces $\{(X_n,d_n,\rho_n)\}_n$ with $0\leq\rho_n\leq 1$ for every n, where the elements of all the spaces are supposed to be in a universe Z. The axioms on the spaces are assumed to be presented uniformly in n; thus $X_n \equiv X(n)$, $\rho_n \equiv \rho(n)$ and $d_n \equiv d(n)$.

The product space is defined as in [6]. In particular,

$$\xi\varepsilon\Pi X_n:\quad \forall m\forall x\forall y(x=y\wedge\xi(m,x)\vdash\xi(m,y))$$
$$\wedge\forall m\exists! x(X(m,x)\wedge\xi(m,x));$$
$$\xi=\eta:\quad \forall m\forall x\forall y(\xi(m,x)\wedge\eta(m,y)\vdash x=y).$$

We may write (x_n) for ξ.

Note. The elementary theory of the product topology can be developed as in [6].

Definition 9.2. $\rho((x_n),(y_n)) = \Sigma\{2^{-n}\rho_n(x_n,y_n);\ n=1,2,\ldots\}$.

Proposition 9.1. 1) ρ is a metric on ΠX_n.

2) The product topology and the metric topology induced by ρ are equivalent.

The mathematical proofs suffice for those properties.

Proposition 9.2. The product space is metric compact if each space is.

Proof. Sequential compactness was established in the general setting (see [6]). As for separability, let $E(m)$ be the separability system of X_m. Define $F(m)$ by:

$$F(\xi):\ \exists n(\forall i\le n(\xi(i)\varepsilon E(i))\wedge\forall i>n(\xi(i)=d_i)).$$

Given (x_m);

$$\forall\varepsilon>0\forall m\exists j_m(\rho_m(x_m,E(m,j_m)<\varepsilon/2).$$

Let N be a natural number such that $1/(2^{N+1}-1) < \varepsilon/2$.

$$\rho((x_m),(E(m,j_m))) = \Sigma\{2^{-m}\rho_m(x_m,E(m,j_m));\ m\le N\}$$
$$+ \Sigma\{2^{-m}\rho_m(x_m,d_m);\ m=N+1,\ldots\}$$
$$\le (\varepsilon/2) + (1/(2^{N+1}-1)) < \varepsilon.$$

(Recall that $\rho_m \le 1$ is assumed.) Thus,

$$\forall\zeta\varepsilon X\forall\varepsilon>0\exists N(F(\xi[n])\wedge\rho(\zeta,\xi[N])<\varepsilon),$$

where

$$\xi[N](m) = \begin{cases} E(m,j_m) & \text{if } m \le N, \\ d_m & \text{if } m > N. \end{cases}$$

F can be enumerated, hence the product is separable by F.

§10 Continuous functions

Here we consider the family of continuous functions on metric spaces, aiming towards the Ascoli-Arzela theorem and the Stone-Weierstrass theorem. Although we must define the distance (pseudo-metric) between functions, we do not define the quotient sets with regards to it. Since mathematically we can follow [1] and [2], let us give informal account of the matter.

Definition 10.1. Consider (X,ρ) and (Y,σ) where metric compactness of X is assumed. Define

$$\rho^*(f,g) = \sup\{\sigma(f(x),g(x));\ x\varepsilon X\},$$

where f and g are maps from X to Y.

Proposition 10.1. 1) $cnt(f,X,Y) \wedge cnt(g,X,Y) \rightarrow R(\rho^*(f,g))$.
2) ρ^* is a pseudo-metric on continuous functions.

Definition 10.2. Let F and δ be parameters.

1) $EC(F,\delta)$: $\forall f\varepsilon F cnt(f,X,Y) \wedge \forall\varepsilon>0(\delta(\varepsilon)>0 \wedge \delta(\varepsilon)$ is a rational)

$\wedge\forall\varepsilon>0\forall f\varepsilon F\forall x\varepsilon X\forall y\varepsilon X(\rho(x,y)<\delta(\varepsilon)\vdash\sigma(f(x),f(y))<\varepsilon)$,

where ε stands for a rational.

(F is an equicontinuous family of functions by δ.)

2) Let F, Φ and μ be parameters.

$$TB^*(F,\Phi,\mu):\ \forall\varepsilon>0\forall n\leq\mu(\varepsilon)\Phi(\varepsilon,n)\varepsilon F$$
$$\wedge\forall\varepsilon>0\forall f\varepsilon F\exists n\leq\mu(\varepsilon)\rho^*(f,\Phi(\varepsilon,n))<\varepsilon.$$

(F is totally bounded by Φ and μ with regards to ρ^*.)

3) $TB'(F,\psi,\nu,x,\lambda)$: $\forall r>0\forall i\leq\lambda(r)\chi(r,i)\varepsilon F$

$\wedge\forall r>0\forall f\varepsilon F\exists i\leq\lambda(r)(\Sigma\{\sigma(f(\psi(n,r)),\chi(r,i,\psi(n,r)));n\leq\nu(r)\}<r)$.

Proposition 10.2. (Ascoli-Arzela).

$$EC(F,\delta) \wedge TB(X,\psi,\nu) \wedge TB'(F,\psi,\nu,\chi,\lambda)$$
$$\rightarrow TB^*(F,\Phi^*,\mu^*)$$

for definable Φ^* and μ^*.

Proof. Define $\mu^*(\varepsilon)$ to be $\lambda(\delta(\varepsilon/3))$ and $\Phi^*(\varepsilon,n)$ to be $\chi(\delta(\varepsilon/3),n)$.

Definition 10.3. $C(X) \equiv C(X,R) : \{f;\ cnt(f,X,R)\}$.

[Stone-Weierstrass theorem] Suppose $G \subset C(X)$. If G is separating, then A(G) is dense in C(X).

This is a mathematical statement in the classical form. The definable interpretation of the theorem will be given in the course of an outline of the proof below.

Definition 10.4. 1) Let δ, g, h be parameters, and let R^+ denote the set of positive reals.

$Sep(G,\delta,g,h)$: $G \subset C(X) \wedge \forall\varepsilon>0(g(\varepsilon)\varepsilon G \wedge h(\varepsilon)\varepsilon G)$

$\wedge\ mp(\delta,R^+,R^+) \wedge \forall\varepsilon>0\forall x\varepsilon X\forall y\varepsilon Y[\rho(x,y)\geq\varepsilon$

$\vdash \forall z\varepsilon X(\rho(x,z)\leq\delta(\varepsilon) \vdash |g(\varepsilon,z)|\leq\varepsilon)$

$\wedge\ \forall z\varepsilon X(\rho(y,z)\leq\delta(\varepsilon) \vdash |g(\varepsilon,z)-1|\leq\varepsilon)]$

$\wedge\ \forall\varepsilon>0\forall y\varepsilon X\forall z\varepsilon X(\rho(y,z)\leq\delta(\varepsilon) \vdash |h(\varepsilon,z)-1|\leq\varepsilon)$.

(G is separating by δ, g and h.)

2) $PL(p,n,N,\phi)$: $\forall j\{j=(i_1,\ldots,i_n) \wedge i_1+\ldots+i_n \leq N$

$\vdash R(\phi(j)) \wedge \forall\alpha(\alpha=(x_1,\ldots,x_n)$

$\vdash \rho(\alpha) = \Sigma\{\phi(j)exp(x_1,j_1)\ldots exp(x_n,j_n);$

$j = (i_1,\ldots,i_n), i_1+\ldots+i_n\leq N\})\}$,

where exp(x,j) expresses the power of x to the j.

3) $\alpha = (x_1,\ldots,x_n)$: $\forall k\leq nR(\alpha(k))\wedge\forall k>n\alpha(k)=0$

4) $\pi(a,k,\alpha,j)$ can be defined in terms of a predicate of the definable inductive definition so that it satisfies:

$$\pi(a,0,\alpha,j) = a,$$

$$\pi(a,k+1,\alpha,j) = \pi(a,k,\alpha,j)\exp(\alpha(k+1),i_{k+1}),$$

where $j = (i_1,\ldots,i_{k+1},\ldots,i_n)$.

5) $\pi(a,\alpha,j) \equiv \pi(a,n,\alpha,j)$.

6) $\phi(j)\exp(x_1,i_1)\ldots\exp(x_n,i_n) \equiv \pi(\phi(j),\alpha,j)$.

7) $SPL(p,n,N,\phi)$: $PL(p,n,N,\phi) \wedge p(0,\ldots,0) = 0$

(p is a strict polynomial of degree N with n variables and with the coefficients determined by ϕ.)

8) $DNS(A(G),\nu,\mu,\Phi_1,\Phi_2,\Phi_3)$:

$$\forall\varepsilon>0\forall f\varepsilon C(X)\{\{\forall i\leq\nu(\varepsilon)\Phi_1(\varepsilon,f,i)\varepsilon G$$
$$\wedge SPL(\Phi_2(\varepsilon,f),\nu(\varepsilon),\mu(\varepsilon),\Phi_3(\varepsilon,f))$$
$$\wedge\forall x\varepsilon X|f(x)-\Phi_2(\varepsilon,f)\circ\Phi_1(\varepsilon,f)(x)|<\varepsilon\},$$

where

$$\Phi_2(\varepsilon,f)\circ\Phi_1(\varepsilon,f)(x)$$
$$\equiv \Phi_2(\varepsilon)(\Phi_1(\varepsilon,f,1)(x),\ldots,\Phi_1(\varepsilon,f,\nu(\varepsilon))(x)).$$

(A(G) is dense in C(X) by ν, μ, Φ_1, Φ_2, Φ_3.)

Note. We do not define A(G) as a set of functions; the A(G) in DNS merely expresses a concept concerning G incorporating with other notions.

9) $H(h,\Psi_1,\Psi_2,\nu,\mu,\Phi_3)$: $h\varepsilon C(X)\wedge\forall n DNS(A(G),\nu,\mu,\Psi_1(n),\Psi_2(n),\Phi_3)$

$\wedge\forall\varepsilon>0\lim\{\|\Psi_2(n,\varepsilon,h)\circ\Psi_1(n,\varepsilon,h)-h\|=0;n=1.2\ldots.\}$,

where $\|f\| = \sup\{|f(x)| : x\varepsilon X\}$.

Proof of the Stone-Weierstrass theorem. It suffices to show that H = C(X). Namely, we can construct definable Ψ_1^*, Ψ_2^*, ν^*, μ^*, Φ_3^* such that

$$\forall h\varepsilon C(X)H(h,\Psi_1^*,\Psi_2^*,\nu^*,\mu^*,\Phi_3^*).$$

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