A note on E. Michael's example and rectangular products

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The notion of a rectangular product was introduced by
B. A. Pasynkov and is used for product theorem in dimension theory.
Here we are concerned especially with a rectangular product
with a metric factor, i.e. $X \times M$ where M is a metric space.
We give a brief introduction to recent results concerning a
rectangular product with a metric factor.

In 1968, Y. Kodama proved the following:

THEOREM (Kodama). If $X \times M$ is normal and countably
paracompact, then $\dim(X \times M) \leqslant \dim X + \dim M$.

Later in 1973, B. A. Pasynkov introduced the notion of a
rectangular product and announced the following interesting theorem.

THEOREM (Pasynkov). If $X \times M$ is rectangular, then
$\dim(X \times M) \leqslant \dim X + \dim M$.

A product space $X \times Y$ is said to be rectangular if
every cozero subset of $X \times Y$ has a σ-locally finite (in $X \times Y$)
covering consisting of cozero rectangles, i.e. products $U \times V$
of cozero subsets of X and Y.

Pasynkov also observed the following:

THEOREM. If $X \times M$ is normal and countably paracompact,
then the space $X \times M$ is rectangular.

Thus Kodama's theorem can also be obtained from Pasynkov's
theorem. The following question naturally arises:

"Conversely, does the rectangularity of $X \times M$ imply normal

and countable paracompactness of X X M?"

The answer is no! For example:

EXAMPLE (Non normal rectangular product). This is suggested by T. Hoshina. For every space X, there exists an extremally disconnected space E(X) called an absolute of X, and a perfect irreducible map E: E(X) ⟶ X. Let X x M be not normal, for example (E. Michael's line) x (Irrationals). Note that E x 1_M: E(X) x M ⟶ X x M is also a perfect map. Since normality is preserved under closed map, E(X) x M is not normal. H. Ohta proved that every product of a metric space and an extremally disconnected space is rectangular. Thus the space E(X) x M is the desired example.

A weaker condition for a product with a metric factor to be rectangular was given by H. Ohta. The condition is a generalization of normal and countable paracompactness. The following diagram will illustrate the situation.

A   boxed[normal and countably paracompact] ⟶ boxed[weak cb]   A'
B   boxed[normal] ⟶ boxed[weak normal]   B'
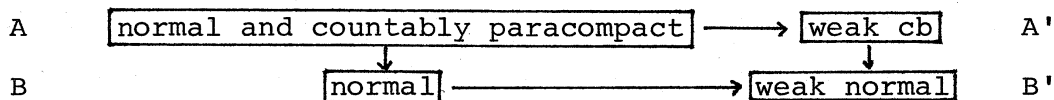
A normal space is <u>countably paracompact</u> iff for any decreasing sequence $\{F_n: n \in N\}$ of closed subsets with empty intersection, there exists a sequence $\{U_n: n \in N\}$ of open subsets with empty intersection such that $F_n \subset U_n$ for all $n \in N$.

A space is <u>weak cb</u> iff for any decreasing sequence $\{F_n: n \in N\}$ of regular closed subsets with empty inteesection,

there exists a sequence $\{U_n: n \in N\}$ of cozero subsets with empty intersection such that $F_n \subset U_n$ for all $n \in N$.

A space is <u>weak normal</u> iff any regular closed subset F and any zero set Z with $F \cap Z = \emptyset$ can be completely separated.

His theorem is as follows:

<u>THEOREM</u>(Ohta). The following conditions for a space X are equivalent.

(1) $X \times M$ is weak cb for any metric space M.

(2) $X \times M$ is weak normal for any metric space M.

(3) $X \times M$ is rectangular for any metric space M.

Sketch of the proof. The equivalence of (1) and (2) follows from the following well known theorem.

<u>THEOREM</u>. Let C be a one point compactification of a countable discrete space, i.e. a converging sequence and the limit point. Then $X \times C$ is $B(B')$ if and only if X is $A(A')$.

The proof of the implication (1) $\longrightarrow$ (3) is similar to the implication (normal and countably paracompact) $\longrightarrow$ (rectangular).

Thus the implication (3) $\longrightarrow$ (1) is the most important. His method is a technic for constructing counterexamples.

So, now, let us mention counterexamples. Only a few non-rectangular products with a metric factor are known. They are as follows:

<u>EXAMPLES</u>(Non-rectangular products with a metric factor).

(1) (Wage's example and Przymusinski's example). $X \times M$ where X is a first countable, separable and Lindelöf space and M is a separable metric space. They showed that $\dim X = \dim M$

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= 0   but   dim X × M > 0.

(2) (Ohta)   If   X   is not weak cb, then there exists a metric space   M   such that the product   X × M   is not rectangular. This result   was used to prove   (3)⟶(1)   of Ohta's theorem mentioned above.

(3) (Tamano)   (Michael's line) × (Irrationals)   is not rectangular.   Let   R   be the set of real numbers,   Q   the set of rational numbers and $P = R \setminus Q$ the set of irrational numbers.   Michael's line is the same as   R   as a set.   The open sets of   X   is the form   U ∨ A, where   U   is an open set of   R   in the usual topology and   A   is a subset of   P.

Sketch of the proof.   Let $D = \{d_n : n \in N\}$ be a countable dense subset of   P   in the usual topology.   Then $Z = \Delta_D = \{(d,d) : d \in D\}$ is a zero set of   X × P.   Then $(X × P) \setminus Z$ is a cozero set and it can be shown that it is not a union of σ-locally finite cozero rectangles.

If a space   X × M   is weak cb, then by Ohta's theorem, the product is rectangular.   Therefore this space   X × P   is not weak cb.   The following question naturally arises:

"Is this space   X × P   weak normal?"

But the answer is no!   In fact $F = Cl_{X × P}(D × P) = (D ∨ Q) × P$ is a regular closed set, $Z' = \Delta_{P \setminus D} = \{(p,p) : p \in P \setminus D\}$ is a zero set and   F   and   Z'   cannot be completely separated.   This fact is obtained by the standard use of Baire's category theorem.

But this fact is obtained more generally by the following remark.

REMARK(Ito). Let M be a non discrete metric space. If X ✗ M is normal, then X is weak cb. So, if X ✗ P = (X ✗ P) ✗ P is weak normal, then X ✗ P is weak cb.

Now, we pose a question. I think the most interesting and important question about this topic is the following one.

QUESTION. If X ✗ M is weak normal (and furthermore M is not discrete), is X ✗ M rectangular (weak cb)?

The following theorem also suggests this question.

THEOREM(Starbird). Let M be a non-discrete metric space. If X ✗ M is normal, then X ✗ M is countably paracompact.

COROLLARY. If X ✗ M is normal, then X ✗ M is rectangular.

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