On products of countable tightness

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If a space \( Y \) has countable tightness, not much can be said about the tightness of \( Y^2 \). We consider what must be true if \( Y^2 \) has countable tightness and \( Y \) is a closed image of some "nice" space \( X \) under a map \( f \). We prove some fairly general theorems concerning the behavior of the map \( f \), and then we apply these results to more special cases.

All our spaces are assumed to be regular and \( T_1 \).

We recall some basic definitions.

A space \( X \) has the weak topology with respect to a cover \( \mathcal{C} \) of (not necessarily closed) subsets, if a subset \( A \) of \( X \) is closed in \( X \) whenever \( A \cap C \) is closed in \( C \) for each \( C \in \mathcal{C} \).

A space \( X \) is a \( k \)-space (resp. sequential space), if \( X \) has the weak topology with respect to its compact (resp. compact metric) subsets. Thus every sequential space is \( k \).

The tightness \( (1) \), \( t(X) \), of a space \( X \) is the least cardinal \( \alpha \) such that whenever \( A \subseteq X \) and \( x \in \overline{A} \), there is a subset \( B \subseteq A \) with \( |B| \leq \alpha \) and \( x \notin \overline{B} \). If \( t(X) \leq \omega \), then
X is said to have countable tightness. It is easy to show that \( t(X) \leq \omega \) if and only if X has the weak topology with respect to its countable subsets. Thus every sequential space has countable tightness.

A space X is strongly collectionwise Hausdorff if whenever \( \{ x_\alpha; \alpha \in A \} \) is a closed discrete subset of X, there exists a discrete collection \( \{ U_\alpha; \alpha \in A \} \) of open subsets such that \( x_\alpha \in U_\alpha \) for each \( \alpha \in A \). Note that every collectionwise normal space is strongly collectionwise Hausdorff.

1. General results. Let \( c \) denote the cardinality of the continuum. A space X is called c-compact, if every subset of A with cardinality c has an accumulation point in X.

**Theorem 1.1.** Suppose that \( f: X \to Y \) is closed, with X strongly collectionwise Hausdorff. If \( t(Y^2) \leq \omega \), then the boundary, \( \partial f^{-1}(y) \), of \( f^{-1}(y) \) is c-compact.

We remark that, if \( Y^2 \) is a k-space with \( t(Y) \leq \omega \), then \( t(Y^2) \leq \omega \). Thus, assuming the continuum hypothesis (CH), we have

**Corollary 1.2.** (CH). Suppose that \( f: X \to Y \) is closed, with X paracompact. Then each \( \partial f^{-1}(y) \) is Lindelöf if either \( Y^2 \) is a k-space with \( t(Y) \leq \omega \), or \( t(Y^2) \leq \omega \).

We don't know whether the CH assumption in Corollary 1.2 can be omitted or not. However, in case where Y is sequential,
we can omit (CH) as will be seen in Corollary 1.4.

**Theorem 1.3.** Suppose that $f: X \to Y$ is closed with $X$ strongly collectionwise Hausdorff and $Y$ is sequential. If $t(Y^2) \leq \omega$, then each $\mathcal{D} f^{-1}(y)$ is $\omega_1$-compact.

**Corollary 1.4.** Suppose that $f: X \to Y$ is closed with $X$ paracompact, $Y$ sequential. If $t(Y^2) \leq \omega$, then each $\mathcal{D} f^{-1}(y)$ is Lindelöf.

The following example shows that the assumption " $Y^2$ is a $k$-space " is not sufficient to obtain " $\mathcal{D} f^{-1}(y)$ is Lindelöf " in Corollary 1.2.

**Example 1.5.** There exists $f: X \to Y$ closed with $X$ locally compact and paracompact, such that $Y^2$ is a $k$-space, but $\mathcal{D} f^{-1}(y)$ is not Lindelöf for some $y \in Y$.

Indeed, for each $\alpha < \omega_1$, let $S(\alpha)$ be a copy of ordinal space $(0, \omega_1)$. Let $X$ be the free union of $\{ S(\alpha); \alpha < \omega_1 \}$. Let $Y$ be the space obtained from $X$ by identifying the point $\omega_1$ in each copy to a single point $\infty$. Let $f: X \to Y$ be the quotient map. Then $X$ is paracompact and locally compact, $f$ is closed, and $f^{-1}(\infty)$ is not Lindelöf. We can prove that $Y^2$ is a $k$-space.

2. Applications. A collection $\mathcal{N}$ of (not necessarily open) subsets of a space $X$ is a $k$-network for $X$ if, whenever $C \subset U$ with $C$ compact and $U$ open, then $C \subset \bigcup \mathcal{F} \subset U$ for some finite subcollection $\mathcal{F}$ of $\mathcal{N}$. An $\mathcal{J}_0^{\omega}$-space is a space with a countable $k$-network, and an $\mathcal{J}^\omega$-space is a space with a $\sigma$-locally finite $k$-network. The concept of $\mathcal{J}_0^{\omega}$-spaces;
$\mathcal{K}$-spaces is introduced by E. Michael (5); P. O'Meara (8).

We say that $X$ is a locally $\mathcal{K}_o$-space if each point of $X$ has a neighborhood which is an $\mathcal{K}_o$-space.

**Theorem 2.1.** (CH). Let $f: X \to Y$ be a closed map.

Let $X$ be a paracompact, locally $\mathcal{K}_o$-space. Then the following are equivalent:

(a) $t(Y^2) \leq \omega$.
(b) each $\mathcal{K}_o f^{-1}(y)$ is Lindelöf.
(c) $Y$ is locally $\mathcal{K}_o$.
(d) $Y$ is locally separable.

**Corollary 2.2.** Let $f: X \to Y$ be a closed map with $X$ locally separable, metric. Then the following are equivalent:

(a) $t(Y^2) \leq \omega$.
(b) each $\mathcal{K}_o f^{-1}(y)$ is Lindelöf.
(c) $Y$ is locally separable.
(d) $Y$ is locally Lindelöf.
(e) $Y$ is an $\mathcal{K}$-space.

A decreasing sequence $(A_n)$ in a space $X$ is a **k-sequence** (7), if $K = \bigcap_{n=1}^{\infty} A_n$ is compact and every neighborhood of $K$ contains some $A_n$. A space $Y$ is a **bi-k-space** (7) if, whenever a filter base $\mathcal{F}$ accumulating at $y \in Y$, then there exists a k-sequence $(A_n)$ in $Y$ such that $y \in F \cap A_n$ for all $n \in \mathbb{N}$ and all $F \in \mathcal{F}$. It is shown that (7) $Y$ is a bi-k-space if and only if $Y$ is a bi-quotient image of a paracompact $\mathbb{N}$-space $X$. Then, by (13), spaces of pointwise countable type are bi-k.
Recall that a space $X$ is a $k_\omega$-space (6), if it has the weak topology with respect to a countable covering of compact subsets of $X$. For a space $Y$, we shall say that $Y$ is a locally $k_\omega$-space, if each point of $Y$ has a neighborhood whose closure is a $k_\omega$-space.

**Theorem 2.3.** (CH). Let $f: X \to Y$ be a closed map with $X$ paracompact bi-$k$. If $t(Y) \leq \omega$, then the following are equivalent. When $Y$ is sequential, the CH assumption can be omitted.

(a) $Y^2$ is a $k$-space.

(b) $Y$ is locally $k_\omega$, or each $\mathcal{g}^{-1}(y)$ is compact.

(c) $Y$ is locally $k_\omega$, or bi-$k$.

**Corollary 2.4.** Let $f: X \to Y$ be a closed map with $X$ or $Y$ sequential. Let $X$ be a paracompact space of pointwise countable type. Then $Y^2$ is sequential if and only if $Y$ is locally $k_\omega$, or bi-$k$.

**Theorem 2.5.** Let $f: X \to Y$ be a closed map with $X$ a paracompact $k^\mathcal{K}$-space. Then $Y^2$ is a $k$-space if and only if $Y$ is metrizable, or $Y$ is an $k^\mathcal{K}$-space which is locally $k_\omega$. 

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References


