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Boundaries of ℓ^2-manifolds in ℓ^2

by

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Let M and N be separable ℓ^2-manifolds with N a Z-set in M. Then N has a collar in M, that is, there exists an open embedding $k : N \times [0,1) \to M$ such that $k(x,0) = x$ for each $x \in N$ (e.g. see [Sa_1] or [Sa_2]). So N may be considered as a "boundary" of M. We call such a submanifold N of M a "Z-submanofold" of M. The following problem was raised by R. D. Anderson [A] (cf. [G] NLC 4).

Problem (A) Under what conditions on the pair (M,N) can M be embedded in ℓ^2 with N as a topological boundary ?

It is seen in [Sa_2] that if M can be embedded in such a way, it can be embedded in ℓ^2 so that N is a bicollared in ℓ^2. In [Sa_3], this embeddeng problem (A) can be reduced to the problem finding a separable complete-metrizable ANR Y such that the attaching space $M \cup_N Y$ is contractible, and a sufficient condition for (A) is given, that is,

Theorem For (A), it is sufficient that M and N contain closed sets A and B respectively such that

(i) A is an AR and B is an ANR,

(ii) $A \cap B$ is a retract of B,

(iii) $M / A \cup B$ is contractible.

Under some homotopical assumptions for M or N , we obtain a necessary and sufficient condition for (A).

1) In the case M is contractible (i.e., $M \cong \ell^2$).

M can be embedded in ℓ^2 as (A) without conditions.

2) In the case M has the homotopy type of S^n $(n = 1, 2, \ldots)$ (i.e., $M \cong S^n \times \ell^2$).

It is necessary and sufficient that N has a component N_0 such that the inclusion $i : N_0 \to M$ induces the epimorphism $i_* : \pi_n(N_0) \to \pi_n(M)$ between the n-th homotopy groups. (Here we assume that each component of N is simply connected if $n > 1$.)

3) In the case N is contractible (i.e., $N \cong \ell^2$).

It is necessary and sufficient that M is also contractible.

4) In the case N has the homotopy type of S^n ($n \geq 2$).

It is necessary sufficient that N contains a deformation retract of M .

The following is obtained in [Sa_2]:

Theorem In order that there exists an embedding $h : M \to \ell^2$ so that $h(N) = \mathrm{bd}\, h(M)$ and $\mathrm{cl}(\ell^2 \setminus h(M))$ is contractible, it is necessary and sufficient that M / N is contractible.

From this result, we can consider the following problem (cf. [Sa_2]):

Problem (B) Assume that M is connected. Under what conditions on the pair (M,N) does there exist an embedding $h : M \to \ell^2$ so thet $h(N) = \mathrm{bd}\, h(M)$ and $\mathrm{cl}(\ell^2 \setminus h(M))$ has the the homotopy type of S^n ?

For this problem (B) in the case $n = 0$, we have an answer in [Sa_2], that is, it is necessary and sufficient that N has just two components N_0 and N_1 and there exists an embedding $\alpha : I \to M$ such that $\alpha(I) \cap N = \{\alpha(0), \alpha(1)\}$, $\alpha(i) \in N_i$ ($i = 0, 1$) and $M / \alpha(I) \cup N$ is contractible

In the case $n \geq 2$, we have an answer in [Sa_3] under the assumption that M and N are simply connected, that is, it is necessary and sufficient that there exists an embedding $\alpha : B^{n+1} \to M$ such that $\alpha(B^{n+1}) \cap N = \alpha(S^n)$, $\alpha(S^n)$ is a retract of N and $M / \alpha(B^{n+1}) \cup N$ is contractible.

Each separable ℓ^2-manifold pair (M,N) with N a Z-submanifold of M is homeomorphic to a pair $(|K| \times \ell^2, |L| \times \ell^2)$ where (K,L) is a locally finite-dimensional countable simplicial complex pair and $|L|$ is collared in $|K|$ ([Sa_3] Theorem 1-3). Then the problem (A) (and (B) resp.) are equivalent to the following problem (A') (and (B') resp.):

<u>Problem (A')</u> (<u>Problem (B')</u> resp.) Under what conditions on the pair (K,L) does exists a locally finite-dimensional countable simplicial complex W such that L is a subcomplex of W and $|K| \underset{|L|}{\cup} |W|$ is contractible (and $|W|$ has the homotopy type of S^n resp.) ?

R e f e r e n c e s

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