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DIFFERENTIAL GRADED MODELS FOR LOCAL RINGS

by

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The purpose of this talk is to describe a method for investigating the Yoneda cohomology algebra $\mathrm{Ext}_R(k,k)$, defined by a commutative noetherian local ring R with maximal ideal $\underline{m}$ and residue field k. Applications include characterizations of local complete intersections and of Golod rings in terms of the algebraic structure of their cohomology, and also (under certain conditions on the characteristic of k) a description of the asymptotic growth of the sequence of Betti numbers $b_i = \dim_k \mathrm{Ext}^i_R(k,k) = \dim_k \mathrm{Tor}^R_i(k,k)$. Our approach is inspired by Sullivan's theory of minimal models for topological spaces [Su]. To a limited extent, namely for rings which are (the localiztion at the irrelevant maximal ideal of) non-negatively graded algebras R' with R'_0 a field of characteristic zero, Sullivan's models can be applied to the problems considered here, as has been recently demonstrated by Félix and Thomas [FT]. However, for general local rings these methods are not available, either because the DG (= differential graded) objects involved are of homological (instead of cohomological) nature, or - much more important - because some of the key theorems of the theory fail already for reasons of characteristic. In our approach we have adopted from rational homotopy theory two essential features: the study of complicated DG algebras via homotopically equivalent "minimal" algebras, and the shifting of the emphasis from the associative homology products to the Lie algebra structure carried by the homotopy groups.

<u>Definition</u>. A graded Lie algebra L over a field k is a collection $\{L^i\}_{i \geqslant 1}$ of finite-dimensional vector spaces, equipped with a bilinear pairing :

$$[\ ,\ ] : L^i \times L^j \longrightarrow L^{i+j} \quad (i \geqslant 1\ ,\ j \geqslant 1\ ),$$

and with a quadratic operator :

$$q\ :\ L^{2i+1} \longrightarrow L^{4i+2} \quad (i \geqslant 0\ ),$$

which satisfy the following conditions (in which we set $|a| = i$ when $a \in L^i$) :

(1) $[a,b] = -(-1)^{|a||b|}[b,a]$;

(2) $(-1)^{|a||c|}[a,[b,c]] + (-1)^{|b||a|}[b,[c,a]] + (-1)^{|c||b|}[c,[a,b]] = 0$;

(3) $[a,[a,a]] = 0$ for $|a|$ odd ;

(4) $[a,a] = 0$ for $|a|$ even ;

(5) $[a,b] = q(a+b) - q(a) - q(b)$ for $|a| = |b|$ odd ;

(6) $q(xa) = x^2 q(a)$ for $x \in k$.

<u>Remarks</u>. (a) The last two conditions imply that for $|a|$ odd one has $[a,a] = 2q(a)$, hence in characteristic 2 $[a,a] = 0$ for all a . On the other hand, this shows that when $\mathrm{char}(k) > 2$, the quadratic operator is defined by the Lie bracket, and the last three conditions are automatically verified. If moreover $\mathrm{char}(k) > 3$, then condition (3) is implied by the "Jacobi identity" (2).

(b) A typical example of graded Lie algebra is provided by the underlying vector space of a graded associative algebra, by setting $[a,b] = ab - (-1)^{|a||b|} ba$, and $q(a) = a^2$. In fact, a universal enveloping algebra for L , denoted UL , exists and is constructed in the usual way. The appropriate version of the Poincaré-Birkhoff-Witt theorem yields the equality of formal power series:

$$\sum_{i \geqslant 0} \dim_k (UL)^i\, t^i = \prod_{i \geqslant 0} (1 - (-t)^{i+1})^{(-1)^i \dim_k L^{i+1}} .$$

The relevance of graded Lie algebras to the theory of cohomology of local rings comes from the fact that $\mathrm{Tor}^R(k,k)$ is a strictly skew-commutative graded Hopf algebra with divided powers. Through a structure theorem for such objects, due to Milnor-Moore ([MM], when $\mathrm{char}(k) = 0$), André ([An], when $\mathrm{char}(k) > 2$), and Sjödin ([Sj], when $\mathrm{char}(k) = 2$), one has the following:

<u>Theorem</u> A. <u>The local ring</u> R <u>defines a graded Lie algebra</u> L_R <u>such that</u> $\mathrm{Ext}_R(k,k)$ <u>is isomorphic, as a Hopf algebra, to the universal enveloping algebra of</u> L_R .

<u>Remarks</u>. (a) In characteristic zero, L_R is canonically isomorphic, as a Lie algebra, to the cohomology $H^*(R,k,k)$ with coefficients in k of the cotangent complex of the R-algebra k (cf. [Q]) .

(b) The coalgebra structure of $\mathrm{Ext}_R(k,k)$ is completely determined by the identification of L_R as a subspace of Ext, while the algebra structure (i.e. the Yoneda products), depends on more complex data, namely the Lie bracket (and the quadratic operator in characteristic 2), of L_R .

<u>Definition</u>. An associative DG k-algebra $V = \{V_i\}_{i \geqslant 0}$ is called a minimal model for R if it satisfies the following conditions:

(1) forgetting the differential, V is free as a skew-commutative k-algebra, i.e. is isomorphic to a tensor product of exterior algebras on generators of odd degree with polynomial algebras on generators of even degree ;

(2) $V_0 = k$;

(3) the differential d of V is of degree -1 and is decomposable, i.e. $d(V) \subset (IV)^2$, $IV = \{V_i\}_{i \geqslant 1}$;

(4) V is homotopically equivalent to the Koszul complex K^R on a minimal set of generators of $\underline{m}$, i.e. there exists a sequence of homomorphisms of DG rings $f_i: U(i-1) \to U(i)$ <u>or</u> $f_i: U(i) \to U(i-1)$, such that $U(0) = K^R$, $U(m) = V$, and $H(f_i)$ is an isomorphism for $i = 1,2,\ldots,m$.

The <u>existence</u> of minimal models can be seen as follows. Choose, by I.S.Cohen's theorem, a regular local ring S with $\dim(S) = \operatorname{edim}(R)$ $(:= \dim_k \underline{m}/\underline{m}^2)$, and a surjection $\eta: S \to \hat{R}$. By the usual procedure of adjoining exterior (resp. polynomial) variables in order to kill cycles of even (resp. odd) degree, and taking care to kill in each degree a minimal set of generators of the corresponding homology S-module, one constructs a DG S-algebra W with the following properties: (1)' W is free as a skew-commutative graded algebra; (2)' $W_0 = S$, and $H(\operatorname{Ker}\eta) = 0$, where $\eta: W \to \hat{R}$ is the obvious extension of the map above; (3)' denoting by IW the kernel of the extension ε to W of the augmentation $\varepsilon: S \to k$, one has $d(IW) \subset (IW)^2$. Now consider the maps of DG rings:

$$(*) \qquad K^R \xrightarrow{\alpha} K^{\hat{R}} \xleftarrow{K^S \otimes \eta} K^S \otimes W \xrightarrow{\varepsilon \otimes W} k \otimes W := V .$$

This is a homotopy equivalence, since α is obtained by completion, while the other two maps induce isomorphisms in homology by the standard fact that $\operatorname{Tor}^S(k,\hat{R})$ can be computed by using resolutions of either argument, or of both arguments. Thus V satisfies codition (4), hence is a minimal model since all the other requirements are fulfilled by construction.

Set $QV = IV/(IV)^2$, and note that choosing a splitting of the map of graded vector spaces $IV \to QV$, we can identify an ordered homogeneous basis $\{x_\alpha\}$ of QV ($|x_\alpha| < |x_\beta|$ implies

$\alpha < \beta$) with a set of generators of the graded k-algebra V . By the decomposability of the differential of V , one has the expressions :

$$d\,x_\gamma = \sum_{|x_\alpha| + |x_\beta| + 1 = |x_\gamma|}^{\alpha < \beta} c^\gamma_{\alpha\beta}\, x_\alpha x_\beta \;+\; \sum_{2|x_\alpha| + 1 = |x_\gamma|} c^\gamma_\alpha\, x^2_\alpha \;+\; y$$

with $y \in (IV)^3$.

Let $L = \{L^i\}$ denote the graded vector space dual of QV , shifted one degree: $L^i = \mathrm{Hom}_k(\,(QV)_{i-1}\,,\,k\,)$ for $i \geqslant 1$, and let $\{\varkappa_\alpha\}$ denote the homogeneous basis dual to the basis $\{x_\alpha\}$. In these notations we can state our main technical result:

<u>Theorem</u> B. <u>Let</u> V <u>be some minimal model for</u> R . <u>Then the formulas</u> :

$$[\varkappa_\alpha, \varkappa_\beta] = -(-1)^{|\varkappa_\alpha||\varkappa_\beta|}[\varkappa_\beta, \varkappa_\alpha] = (-1)^{|\varkappa_\alpha|-1} \sum_\gamma c^\gamma_{\alpha\beta}\varkappa_\gamma \;,\; \underline{\text{for}}\ \alpha<\beta;$$

$$q(\varkappa_\alpha) = \sum_\gamma c^\gamma_\alpha \varkappa_\gamma \;,\; \underline{\text{for}}\ |\varkappa_\alpha|\ \underline{\text{odd}}\;,$$

<u>define on</u> L <u>a structure of graded Lie algebra</u>.

<u>Moreover</u>, $L \simeq L_R[2]$ <u>as graded Lie algebras</u>, <u>where</u> $L_R[2] = \{L^i_R\}_{i \geqslant 2}$.

<u>Example</u>. Recall that R is called a complete intersection if $\mathrm{Ker}(\eta : S \to \hat{R})$ is generated by an S-regular sequence. Since one can choose W to be the Koszul complex on that sequence, (*) above shows that a minimal model for R is given by the exterior algebra, equipped with the trivial differential, of the vector space $H_1(K^R)$. The well-known fact that $L_R[2]$ is concentrated in degree 2 follows immediately; in particular, $L_R[2]$ is abelian, i.e. has trivial bracket and quadratic operator. Moreover, one easily reads off from the expression for [,] , that the elements of L^2_R commute if and only if the ring R has the property that $\Lambda^2 H_1(K^R) \to H_2(K^R)$ is a monomorphism.

In general, there exist <u>coefficientwise</u> inequalities of formal power series :

$$(**)\quad \frac{(1+t)^{\mathrm{edim}(R)}}{(1-t^2)^{\mathrm{edim}(R)-\dim(R)}} \leqslant \sum b_i t^i \leqslant \frac{(1+t)^{\mathrm{edim}(R)}}{1-\sum \dim_k H_i(K^R)t^{i+1}}$$

in which the lower bound is attained precisely by the complete intersections. Rings for which the upper bound holds are called Golod rings; examples are given by homomorphic images of regular local rings modulo ideals generated by the maximal minors of $r \times s$ matrices, when such ideals have the maximal possible grade (= $s-r+1$), and the rank of the corresponding matrix over the residue field is at most $r-2$. The next result characterizes the cases when some equality holds in (**) in terms of Yoneda products.

<u>Theorem</u> C. (i) R <u>is a complete intersection if and only if</u> $L_R[2]$ <u>is an abelian graded Lie algebra</u>.

(ii) R <u>is a Golod ring if and only if</u> $L_R[2]$ <u>is a free graded Lie algebra</u>.

We now turn to the study of the asymptotic behaviour of the non-decreasing, for $i > \dim(R)$, sequence of Betti numbers b_i . In an important recent paper [FH], Félix and Halperin have determined the asymptotic growth of the sequence $\{\sum_{j \leqslant i} \dim_{\mathbb{Q}} H^j(\Omega X, \mathbb{Q})\}_{i \geqslant 0}$, defined by a simply-connected finite CW complex X . Using some ideas from their work, and expressing the homology of some homomorphic images of V in terms of the homology of subalgebras of the minimal R-free resolution of k , we are able to prove:

<u>Theorem</u> D. <u>Let</u> $\{b_i\}_{i \geqslant 0}$ <u>be the sequence of Betti numbers of the local ring</u> $(R,\underline{m},k)$. <u>Suppose that</u> $\mathrm{char}(k)=0$, <u>or that</u> $\mathrm{char}(k)-1 > \min(\mathrm{edim}(R)-\mathrm{depth}(R), s)$, <u>where</u> s <u>is defined</u>

to be the largest integer such that $m^s \neq 0$ when R is artinian, and infinity otherwise.

Then there exists an integer N such that for $i > N$:

- either b_i is a polynomial in i (this characterizes complete intersections, and in this case the polynomial, explicitely given by (**) , is of degree $\operatorname{edim}(R) - \dim(R) - 1$);

- or there exist real numbers $C > 1$, $D > 1$, such that $C^i \leqslant b_i < D^i$.

Remarks. (a) In case when R is the localization at the irrelevant maximal ideal of a graded algebra over a field of characteristic zero, this result has been obtained by Félix and Thomas, essentially by reducing the algebraic problem to a topological one (more precisely: by introducing new gradings in such a way as to make the DG algebras arising in the present context compatible with those considered in Sullivan's theory).

(b) It has been conjectured in [Av_1] that the conclusion of the Theorem holds without restrictions on the characteristic. Further evidence for this is given by Theorem F below.

As an immediate corollary to the previous result, one obtains the validity of a conjecture of Golod and Gulliksen, bearing on the analytic properties of the Poincaré series $P_R(t) = \sum b_i t^i$. Note that by the upper bound in (**), this formal power series always represents the development around the origin of an analytic function.

Theorem E. Assume the restrictions on $\operatorname{char}(k)$ introduced in Theorem D , and denote by r_R the radius of convergence of the Poincaré series $P_R(t)$. Then the following hold:

(i) $r_R = \infty$ if and only if R is regular ;

(ii) $r_R = 1$ if and only if R is a non-regular complete intersection ;

(iii) $0 < r_R < 1$ <u>otherwise</u> .

It is relevant to note at this point that the conclusions of theorems D and E hold for rings having rational Poincaré series. However, disproving an old conjecture, Anick [Ak] has demonstrated that not all local rings have this property. A construction of rings with transcendental $P_R(t)$, which is more in the line of this talk, i.e. is carried out using graded Lie algebras, has subsequently been given by Löfwall and Roos [LR].

Finally let us mention the following still open conjecture, discussed in [Av_2] , whose validity would imply theorems D and E :

<u>Conjecture</u>. Either R is a complete intersection, or L_R contains a non-abelian free Lie subalgebra.

Apart from many examples, where the statement is known to hold (e.g. for all homomorphic images of regular local rings modulo monomials in some regular sequence, cf. [Av_2]), the following result is available:

<u>Theorem</u> F. <u>The conjecture holds if either</u> $\operatorname{edim}(R) - \operatorname{depth}(R) \leqslant 3$, <u>or</u> R <u>is Gorenstein with</u> $\operatorname{edim}(R) - \dim(R) = 4$.

The proof of the last theorem is contained in [Av_2], where a sufficient condition for L_R to contain a free graded Lie subalgebra is given in terms of the Massey products structure of the homology of "partial resolutions" (e.g. the Koszul complex K^R). The proofs of theorems C, D, and E will be published elsewhere.

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