Local Rings with Multiplicity Two (Commutative Algebra and Algebraic Geometry)

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Local rings with multiplicity two

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Let \((A,\mathfrak{m},k)\) be a Noetherian local ring and let \(e(A)\) be the multiplicity of \(A\). It is well known that \(A\) is regular if and only if \(A\) is unmixed and \(e(A) = 1\). But, in general, a local ring with multiplicity 2 is not a hypersurface even if it is unmixed.

Example. Let \(k\) be a field, \(d \geq 2\) an integer and \(X_1, \ldots, X_d, Y_1, \ldots, Y_d\) indeterminates over \(k\). We put
\[
A = k[[X_1, \ldots, X_d, Y_1, \ldots, Y_d]]/(X_1, \ldots, X_d) \cap (Y_1, \ldots, Y_d).
\]
Then, \(A\) is unmixed and \(e(A) = 2\), but \(A\) is not a hypersurface.

Note that \(A\) does not satisfy \((S_2)\).

In a recent work [1], S. Goto studied Buchsbaum rings with multiplicity 2. Inspired by [1], K. Watanabe raised the following questions.

1. Is a local ring with multiplicity 2 satisfying \((S_2)\) a hypersurface?
2. Is a local ring with multiplicity 3 satisfying \((S_2)\) Cohen-Macaulay?

In this note we give an affirmative answer to the question (1) under some additional conditions and we give a counter example to the question (2).

Throughout this note a ring means a commutative Noetherian ring with a unit.
First we recall basic properties of the multiplicity of local rings.

Let \((A, \mathfrak{m}, k)\) be a local ring. We put
\[
\text{Assh}(A) = \left\{ p \in \text{Ass}(A) \mid \dim A/p = \dim A \right\}.
\]
The following result can be found in [2].

Proposition 1.
\begin{align*}
(1) \quad e(A) &= \sum_{p \in \text{Assh}(A)} l(A_p)e(A/p) \\
(2) \quad &\text{Let } p \in \text{Spec}(A). \text{ If } \text{ht } p + \dim A/p = \dim A \text{ and } A/p \text{ is analytically unramified, then } e(A_p) \leq e(A).
\end{align*}

The notion of ideal transform plays an important rôle in the sequel.

We recall the definition: let \(R\) be a ring, \(I\) an ideal of \(R\) and \(M\) a finitely generated \(R\)-module; we define
\[
D_I(M) = \lim_{\longrightarrow} \hom_R(I^n, M)
\]
and call it the ideal transform of \(M\) with respect to \(I\).

Proposition 2. Let \(R, I\) and \(M\) be as above. Then,
\begin{align*}
(1) \quad H^i_I(D_I(M)) &= \begin{cases} 
0 & \text{for } i \leq 1 \\
H^i_I(M) & \text{for } i \geq 2
\end{cases} \\
(2) \quad &\text{we have the following exact sequence}
0 \longrightarrow H^0_I(M) \longrightarrow M \longrightarrow D_I(M) \longrightarrow H^1_I(M) \longrightarrow 0
\end{align*}
and
\begin{itemize}
\item[(3)] \(D_I(M)_p = M_p\) for \(p \notin V(I)\).
\end{itemize}
We need a result of M. Hochster, the "direct summand conjecture" (cf. [3]).

Proposition 3. Let $R$ be a regular ring containing a field and let $S$ be a module-finite extension algebra of $R$. Then, $R$ is a direct summand of $S$ as an $R$-module.

2. Local rings with multiplicity 2.

First we give an affirmative answer to the question (1) under the condition that the local ring is complete and contains a field.

Theorem 4. Let $(A, m, k)$ be a complete local ring containing a field. Assume that $A$ satisfies $(S_2)$ and $e(A) = 2$. Then $A$ is a hypersurface with multiplicity 2.

(Proof). It is sufficient to prove that $A$ is Cohen-macaulay. If $\dim A \leq 2$ there is nothing to prove. We will prove the assertion by induction on $\dim A$. It is easy to see that $e(A_p) \leq e(A)$ for all $p \in \text{Spec}(A)$. By the induction hypothesis we may assume that $A_p$ is Cohen-Macaulay for $p \in \text{Spec}(A) - \{m\}$. In particular, we may assume that $\_1(H^i_m(A)) < \infty$ for $0 \leq i < \dim A$. Assume that $\dim A = 3$. We may assume that $k$ is an infinite field, so that there exists an S.O.P. $a_1, a_2, a_3$ such that $m^n = (a_1, a_2, a_3)m^{n-1}$ for some positive integer $n$. Set $S = k[[a_1, a_2, a_3]]$. Then $S$ is a regular local ring and $A$ is a module finite extension of $S$. Using Proposition 3, we get an exact sequence.
where \( M = (a_{ij}) \) is an \((n-1) \times n\) matrix with \( a_{ij} \in \mathbb{R} \) and \( \mathbb{R} \) is the maximal ideal of \( S \). We want to show that \( A \) is Cohen-Macaulay.

Assume the contrary. Since \((A/S)_p\) is free for \( p \in \text{Spec}(S) - \{ \mathbb{R} \}\), the ideal generated by the maximal minors of \( M \) is an \( \mathbb{R} \)-primary ideal of height at most 2. This is a contradiction. Let \( \dim A \geq 4 \).

Choose a non zero divisor \( x \) such that \( e(A/xA) = 2 \). We have an exact sequence

\[
0 \longrightarrow A/xA \longrightarrow D_m(A/xA) \longrightarrow H_{\mathfrak{m}}^1(A/xA) \longrightarrow 0.
\]

The ideal transform \( D_m(A/xA) \) is a finite product of complete local rings with multiplicity two and satisfies \((S_2)\) by Proposition 2. Hence \( D_m(A/xA) \) is Cohen-Macaulay by the induction hypothesis. It is easy to see that \( H_{\mathfrak{m}}^i(A/xA) = (0) \) for \( 2 \leq i < \dim A/xA \). From the exact sequence

\[
0 \longrightarrow A \longrightarrow A \longrightarrow A/xA \longrightarrow 0
\]

we get the exact sequence

\[
0 \longrightarrow H_{\mathfrak{m}}^1(A/xA) \longrightarrow H_{\mathfrak{m}}^2(A) \longrightarrow H_{\mathfrak{m}}^2(A) \longrightarrow 0.
\]

Since \( \mathbb{R}(H_{\mathfrak{m}}^2(A)) < \infty \), we have \( H_{\mathfrak{m}}^2(A) = (0) \) by Nakayama's lemma.

Thus, \( A \) is Cohen-Macaulay as required.

For local rings not containing a field we have the following result.

Theorem 5. Let \((A, \mathfrak{m}, k)\) be a complete local ring which is not a domain. Assume that \( e(A) = 2 \) and \( A \) satisfies \((S_2)\). Then,
A is a hypersurface.

The following result is the main theorem of [1].

Corollary 6. Let \((A, m, k)\) be a Buchsbaum ring with \(\dim A \geq 2\) and \(e(A) = 2\). Then, \(H^i_m(A) = (0)\) for \(2 \leq i < \dim A\) and \(\ell(H^1_m(A)) \leq 1\).

If \(A\) contains a field we can give a simple proof of this result by Theorem 4. Another consequence of Theorem 4 is:

Corollary 7. Let \(R\) be a regular local ring containing a field and let \(I\) be an ideal of \(R\) such that \(e(R/I) = 2\) and \(\text{pd}_{R/I}I/1^2 < \infty\). Then \(I\) is generated by an \(R\)-sequence.

Example. Let \(k\) be a field and let \(X_1, X_2, X_3, Y_1, Y_2, Y_3\) be indeterminates over \(k\). We put

\[
A = k[[X_1, X_2, X_3, Y_1, Y_2, Y_3]]/(X_1Y_1 + X_2Y_2 + X_3Y_3, (Y_1, Y_2, Y_3)^2).
\]

Then \(A\) satisfies \((S_2)\) and \(e(A) = 2\). But \(A\) is not Cohen-Macaulay.

During the symposium C. Huneke and S. Goto communicated to me the following generalization of Theorem 4.

Theorem. Let \(A\) be a complete local ring containing a field. Assume that
(1) $A$ satisfies $(S_n)$, $n \leq \dim A$

(2) $e(A) \leq n$.

Then $A$ is Cohen-Macaulay.

Reference