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Local rings with multiplicity two

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Let \((A, \mathfrak{m}, k)\) be a Noetherian local ring and let \(e(A)\) be the multiplicity of \(A\). It is well known that \(A\) is regular if and only if \(A\) is unmixed and \(e(A) = 1\). But, in general, a local ring with multiplicity 2 is not a hypersurface even if it is unmixed.

Example. Let \(k\) be a field, \(d \geq 2\) an integer and \(X_1, \ldots, X_d, Y_1, \ldots, Y_d\) indeterminates over \(k\). We put
\[
A = k[[X_1, \ldots, X_d, Y_1, \ldots, Y_d]]/(X_1, \ldots, X_d) \cap (Y_1, \ldots, Y_d).
\]
Then, \(A\) is unmixed and \(e(A) = 2\), but \(A\) is not a hypersurface. Note that \(A\) does not satisfy \((S_d)\).

In a recent work [1], S. Goto studied Buchsbaum rings with multiplicity 2. Inspired by [1], K. Watanabe raised the following questions.

1. Is a local ring with multiplicity 2 satisfying \((S_d)\) a hypersurface?
2. Is a local ring with multiplicity 3 satisfying \((S_d)\) Cohen-Macaulay?

In this note we give an affirmative answer to the question (1) under some additional conditions and we give a counter example to the question (2).

Throughout this note a ring means a commutative Noetherian ring with a unit.

1. Preliminaries.
First we recall basic properties of the multiplicity of local rings. Let \((A, \mathfrak{m}, k)\) be a local ring. We put
\[
\text{Assh}(A) = \left\{ p \in \text{Ass}(A) \mid \dim A/p = \dim A \right\}.
\]
The following result can be found in [2].

**Proposition 1.**

1. \(e(A) = \sum_{p \in \text{Assh}(A)} \frac{1}{\ell(A/p)} e(A/p)\)

2. Let \(p \in \text{Spec}(A)\). If \(h \text{t } p + \dim A/p = \dim A\) and \(A/p\) is analytically unramified, then \(e(A/p) \leq e(A)\).

The notion of ideal transform plays an important rôle in the sequel. We recall the definition: let \(R\) be a ring, \(I\) an ideal of \(R\) and \(M\) a finitely generated \(R\)-module; we define
\[
D_I(M) = \lim_{\rightarrow n} \text{Hom}_R(I^n, M)
\]
and call it the ideal transform of \(M\) with respect to \(I\).

**Proposition 2.** Let \(R, I\) and \(M\) be as above. Then,

1. \(H^i_I(D_I(M)) = \begin{cases} 
0 & \text{for } i \leq 1 \\
H^i_I(M) & \text{for } i \geq 2 
\end{cases}\)

2. we have the following exact sequence
\[
0 \rightarrow H^0_I(M) \rightarrow M \rightarrow D_I(M) \rightarrow H^1_I(M) \rightarrow 0
\]
and

3. \(D_I(M)_p = M_p\) for \(p \notin V(I)\).
We need a result of M. Hochster, the "direct summand conjecture" (cf. [3]).

Proposition 3. Let $R$ be a regular ring containing a field and let $S$ be a module-finite extension algebra of $R$. Then, $R$ is a direct summand of $S$ as an $R$-module.

2. Local rings with multiplicity 2.

First we give an affirmative answer to the question (1) under the condition that the local ring is complete and contains a field.

Theorem 4. Let $(A, m, k)$ be a complete local ring containing a field. Assume that $A$ satisfies $(S_2)$ and $e(A) = 2$. Then $A$ is a hypersurface with multiplicity 2.

(Proof). It is sufficient to prove that $A$ is Cohen-macaulay. If $\dim A \leq 2$ there is nothing to prove. We will prove the assertion by induction on $\dim A$. It is easy to see that $e(A_p) \leq e(A)$ for all $p \in \text{Spec}(A)$. By the induction hypothesis we may assume that $A_p$ is Cohen-Macaulay for $p \in \text{Spec}(A) - \{m\}$. In particular, we may assume that $\dim^i_H(A) < \infty$ for $0 \leq i < \dim A$. Assume that $\dim A = 3$. We may assume that $k$ is an infinite field, so that there exists an S.O.P. $a_1, a_2, a_3$ such that $m^n = (a_1, a_2, a_3)m^{n-1}$ for some positive integer $n$. Set $S = k[[a_1, a_2, a_3]]$. Then $S$ is a regular local ring and $A$ is a module finite extension of $S$. Using Proposition 3, we get an exact sequence.
where $M = (a_{ij})$ is an $(n-1) \times n$ matrix with $a_{ij} \in \mathfrak{n}$ and $\mathfrak{n}$ is the maximal ideal of $S$. We want to show that $A$ is Cohen-Macaulay.

Assume the contrary. Since $(A/S)_p$ is free for $p \in \text{Spec}(S) - \{ \mathfrak{n} \}$ the ideal generated by the maximal minors of $M$ is an $\mathfrak{n}$-primary ideal of height at most 2. This is a contradiction. Let $\dim A \geq 4$.

Choose a non zero divisor $x$ such that $e(A/xA) = 2$. We have an exact sequence

$$0 \longrightarrow A/xA \longrightarrow D_m(A/xA) \longrightarrow H^1_m(A/xA) \longrightarrow 0.$$  

The ideal transform $D_m(A/xA)$ is a finite product of complete local rings with multiplicity two and satisfies $(S_2)$ by Proposition 2. Hence $D_m(A/xA)$ is Cohen-Macaulay by the induction hypothesis. It is easy to see that $H^i_m(A/xA) = (0)$ for $2 \leq i < \dim A/xA$.

From the exact sequence

$$0 \longrightarrow A \longrightarrow A \longrightarrow A/xA \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow H^1_m(A/xA) \longrightarrow H^2_m(A) \xrightarrow{x} H^2_m(A) \longrightarrow 0.$$  

Since $1(H^2_m(A)) < \infty$, we have $H^2_m(A) = (0)$ by Nakayama's lemma.

Thus, $A$ is Cohen-Macaulay as required.

For local rings not containing a field we have the following result.

**Theorem 5.** Let $(A,m,k)$ be a complete local ring which is not a domain. Assume that $e(A) = 2$ and $A$ satisfies $(S_2)$. Then,
A is a hypersurface.

The following result is the main theorem of [1].

Corollary 6. Let \((A, m, k)\) be a Buchsbaum ring with \(\dim A \geq 2\) and \(e(A) = 2\). Then, \(\hat{H}^i_{m}(A) = 0\) for \(2 \leq i < \dim A\) and \(\hat{H}^i_{m}(A) \leq 1\).

If \(A\) contains a field we can give a simple proof of this result by Theorem 4. Another consequence of Theorem 4 is:

Corollary 7. Let \(R\) be a regular local ring containing a field and let \(I\) be an ideal of \(R\) such that \(e(R/I) = 2\) and \(pd_{R/I} I^{1/2} < \infty\). Then \(I\) is generated by an \(R\)-sequence.

Example. Let \(k\) be a field and let \(X_1 X_2 X_3 Y_1 Y_2 Y_3\) be indeterminates over \(k\). We put

\[ A = k[[X_1, X_2, X_3, Y_1, Y_2, Y_3]]/(X_1 Y_1 + X_2 Y_2 + X_3 Y_3, (Y_1, Y_2, Y_3)^2). \]

Then \(A\) satisfies \((S_2)\) and \(e(A) = 2\). But \(A\) is not Cohen-Macaulay

During the symposium C. Huneke and S. Goto communicated to me the following generalization of Theorem 4.

Theorem. Let \(A\) be a complete local ring containing a field. Assume that
(1) $A$ satisfies $(S_n)$, \( n \leq \dim A \)

(2) \( e(A) \leq n \).

Then $A$ is Cohen-Macaulay.

Reference

