Local rings with multiplicity two

Shin Ikeda  
(Nagoya University)

Let \((A, \mathfrak{m}, k)\) be a Noetherian local ring and let \(e(A)\) be the multiplicity of \(A\). It is well known that \(A\) is regular if and only if \(A\) is unmixed and \(e(A) = 1\). But, in general, a local ring with multiplicity 2 is not a hypersurface even if it is unmixed.

Example. Let \(k\) be a field, \(d \geq 2\) an integer and \(X_1, \ldots, X_d, Y_1, \ldots, Y_d\) indeterminates over \(k\). We put
\[
A = k[[X_1, \ldots, X_d, Y_1, \ldots, Y_d]]/(X_1, \ldots, X_d) \cap (Y_1, \ldots, Y_d).
\]
Then, \(A\) is unmixed and \(e(A) = 2\), but \(A\) is not a hypersurface. Note that \(A\) does not satisfy \((S_2)\).

In a recent work [1], S. Goto studied Buchsbaum rings with multiplicity 2. Inspired by [1], K. Watanabe raised the following questions.

(1) Is a local ring with multiplicity 2 satisfying \((S_2)\) a hypersurface?
(2) Is a local ring with multiplicity 3 satisfying \((S_2)\) Cohen-Macaulay?

In this note we give an affirmative answer to the question (1) under some additional coditions and we give a counter example to the question (2).

Throughout this note a ring means a commutative Noetherian ring with a unit.

1. Preliminaries.
First we recall basic properties of the multiplicity of local rings. Let \((A, \mathfrak{m}, k)\) be a local ring. We put
\[
\text{Assh}(A) = \left\{ p \in \text{Ass}(A) \mid \dim A/p = \dim A \right\}.
\]
The following result can be found in [2].

Proposition 1.

(1) \[ e(A) = \sum_{p \in \text{Assh}(A)} l(A_p) e(A/p) \]

(2) Let \( p \in \text{Spec}(A) \). If \( \text{ht } p + \dim A/p = \dim A \) and \( A/p \) is analytically unramified, then \( e(A_p) \leq e(A) \).

The notion of ideal transform plays an important rôle in the sequel. We recall the definition: let \( R \) be a ring, \( I \) an ideal of \( R \) and \( M \) a finitely generated \( R \)-module; we define
\[
D_I(M) = \lim_{\to n} \text{Hom}_R(I^n, M)
\]
and call it the ideal transform of \( M \) with respect to \( I \).

Proposition 2. Let \( R, I \) and \( M \) be as above. Then,

(1) \[ H^i_I(D_I(M)) = \begin{cases} (0) & \text{for } i \leq 1 \\ H^i_I(M) & \text{for } i \geq 2 \end{cases} \]

(2) we have the following exact sequence
\[
0 \to H^0_I(M) \to M \to D_I(M) \to H^1_I(M) \to 0
\]
and

(3) \[ D_I(M)_p = M_p \quad \text{for} \quad p \notin V(I). \]
We need a result of M. Hochster, the "direct summand conjecture" (cf. [3]).

Proposition 3. Let \( R \) be a regular ring containing a field and let \( S \) be a module-finite extension algebra of \( R \). Then, \( R \) is a direct summand of \( S \) as an \( R \)-module.

2. Local rings with multiplicity 2.

First we give an affirmative answer to the question (1) under the condition that the local ring is complete and contains a field.

Theorem 4. Let \((A,\mathfrak{m},k)\) be a complete local ring containing a field. Assume that \( A \) satisfies \((S_2)\) and \( e(A) = 2 \). Then \( A \) is a hypersurface with multiplicity 2.

(Proof). It is sufficient to prove that \( A \) is Cohen-macaulay. If \( \dim A \leq 2 \) there is nothing to prove. We will prove the assertion by induction on \( \dim A \). It is easy to see that \( e(A_p) \leq e(A) \) for all \( p \in \text{Spec}(A) \). By the induction hypothesis we may assume that \( A_p \) is Cohen-Macaulay for \( p \in \text{Spec}(A) - \{ \mathfrak{m} \} \). In particular, we may assume that \( \dim H^i_{\mathfrak{m}}(A) < \infty \) for \( 0 \leq i < \dim A \). Assume that \( \dim A = 3 \). We may assume that \( k \) is an infinite field, so that there exists an S.O.P. \( a_1, a_2, a_3 \) such that \( \mathfrak{m}^n = (a_1, a_2, a_3)\mathfrak{m}^{n-1} \) for some positive integer \( n \). Set \( S = k[[a_1, a_2, a_3]] \). Then \( S \) is a regular local ring and \( A \) is a module finite extension of \( S \). Using Proposition 3, we get an exact sequence
0 \rightarrow S^{n-1} \rightarrow S^n \rightarrow A/S \rightarrow 0,

where \( M = (a_{ij}) \) is an \((n-1) \times n\) matrix with \( a_{ij} \in \frac{n}{n} \) and \( n \) is the maximal ideal of \( S \). We want to show that \( A \) is Cohen–Macaulay.

Assume the contrary. Since \( (A/S)_p \) is free for \( p \in \text{Spec}(S) - \{ \frac{n}{n} \} \)
the ideal generated by the maximal minors of \( M \) is an \( n \)-primary ideal of height at most 2. This is a contradiction. Let \( \dim A \geq 4 \).

Choose a non-zero divisor \( x \) such that \( e(A/xA) = 2 \). We have an exact sequence

\[ 0 \rightarrow A/xA \rightarrow D_m(A/xA) \rightarrow H^1_m(A/xA) \rightarrow 0. \]

The ideal transform \( D_m(A/xA) \) is a finite product of complete local rings with multiplicity two and satisfies \((S_2)\) by Proposition 2. Hence \( D_m(A/xA) \)
is Cohen–Macaulay by the induction hypothesis. It is easy to see that

\[ H^i_m(A/xA) = (0) \quad \text{for} \quad 2 \leq i < \dim A/xA. \]

From the exact sequence

\[ 0 \rightarrow A \rightarrow A \rightarrow A/xA \rightarrow 0 \]

we get the exact sequence

\[ 0 \rightarrow H^1_m(A/xA) \rightarrow H^2_m(A) \rightarrow H^2_m(A) \rightarrow 0. \]

Since \( \frac{1}{1(\frac{H^2_m(A)}{\infty})} \), we have \( H^2_m(A) = (0) \) by Nakayama's lemma.

Thus, \( A \) is Cohen–Macaulay as required.

For local rings not containing a field we have the following result.

Theorem 5. Let \( (A, m, k) \) be a complete local ring which is not a domain. Assume that \( e(A) = 2 \) and \( A \) satisfies \((S_2)\). Then,
A is a hypersurface.

The following result is the main theorem of [1].

Corollary 6. Let \((A, m, k)\) be a Buchsbaum ring with \(\dim A \geq 2\) and \(e(A) = 2\). Then, \(H^i_m(A) = (0)\) for \(2 \leq i < \dim A\) and \(\text{ht}(H^1_m(A)) \leq 1\).

If \(A\) contains a field we can give a simple proof of this result by Theorem 4. Another consequence of Theorem 4 is:

Corollary 7. Let \(R\) be a regular local ring containing a field and let \(I\) be an ideal of \(R\) such that \(e(R/I) = 2\) and \(\text{pd}_{R/I} I^2 < \infty\). Then \(I\) is generated by an \(R\)-sequence.

Example. Let \(k\) be a field and let \(X_1, X_2, X_3, Y_1, Y_2, Y_3\) be indeterminates over \(k\). We put

\[ A = k[[X_1, X_2, X_3, Y_1, Y_2, Y_3]]/(X_1Y_1 + X_2Y_2 + X_3Y_3, (Y_1Y_2Y_3)^2). \]

Then \(A\) satisfies (\(S_2\)) and \(e(A) = 2\). But \(A\) is not Cohen-Macaulay.

During the symposium C. Huneke and S. Goto communicated to me the following generalization of Theorem 4.

Theorem. Let \(A\) be a complete local ring containing a field. Assume that
(1) \( A \) satisfies \( (S_n) \), \( n \leq \text{dim} \ A \)

(2) \( e(A) \leq n \).

Then \( A \) is Cohen-Macaulay.

Reference