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THE KOSZUL COMPLEX OF BUCHSBAUM MODULES

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Let $(A,\mathfrak{M},k)$ be a Noetherian local ring and M be a finite A-module of dimension d. For simplicity we assume that A is complete. An element x in $\mathfrak{M}$ is called a parameter for M if $\dim(M/xM)<\dim(M)$. A system of elements in $\mathfrak{M}$ is refered to as a sub-system of parameters for M (s.s.o.p.), if it can be extended to a s.o.p. for M. $L_A(\ )$ denotes the length of an A-module, $h^i(\ )$ the length (or the dimension as a vector space) of the i-th local cohomology module $H^i_{\mathfrak{M}}(\ )$ and $h_i(\underline{x};M)$ the length of the Koszul homology module $H_i(\underline{x};M)$.

§1. The Koszul Homology and the local cohomology.

We start with the definition and some basic properties of Buchsbaum (abbreviated to Bbm) modules.

DEFINITION. (i) $x\in\mathfrak{M}$ is said to be weakly M-regular if

$$\mathfrak{M}(0:_M x) = 0.$$

(ii) $\underline{x} = \{x_1,\dots,x_r\}$ is called a weak M-sequence if x_i is weakly $M/(x_1,\dots,x_{i-1})M$ -regular for $i=1,\dots,r$.

(iii) M is called a Buchsbaum module if any s.o.p. for M is a weak M-sequence.

__THEOREM__.(Stückrad, Vogel) The following are equivalent.

(i) M is a Bbm module.

(ii) For any s.o.p. $\underline{x}$ for M, the difference $L_A(M/(\underline{x})M)-e_0(\underline{x};M)$ is an invariant I(M) not depending on $\underline{x}$.

(iii) The natural limit map $H^i(\mathfrak{m};M) \longrightarrow H^i_{\mathfrak{m}}(M)$ is surjective for all $i < d=\dim(M)$.

From the historical point of view, the second charactrization seems to be the most meaningful, because the theory of Bbm-module has its origin in the following words of Buchsbaum: "It would of course be hoped that the difference between the length and multiplicity could be determined by the difference dim(R) - codim(R) and/or __other invariants yet to be found__." On the other hand, the third one implies that the local cohomology must play a role to be respected as well as the Koszul homology. Indeed in section 2 a new relation between the local cohomologies and the multiplicity will be stated.

Our first result comes from the following observation.

__COROLLARY__. If M is a Bbm module then $\mathfrak{m}H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$.

The converse is not true. We call such a module satisfying the conclusion of the corollary a __quasi-Buchsbaum module__. We must make the difference of the modules clearer.

<u>THEOREM</u>(1.1). Let $x \in \mathfrak{m}^2$ be a parameter for M. If M is quasi-Buchsbaum, then so is M/xM.

If besides $L_A((0:x)_M) < \infty$, then the converse is also true.

Proof. We must prove that $\mathfrak{m}H^i_{\mathfrak{m}}(M/xM)=0$ for i=0,...,d-2. To begin with, note that $\mathfrak{m}^2 H^i_{\mathfrak{m}}(M/xM) = 0$ for all $i<d-1$. $M':=M/xM$. Set $\mathfrak{a}=(0 : H^0_{\mathfrak{m}}(M'))$ and suppose $\mathfrak{a} \subsetneq \mathfrak{m}$. There exists $z \in \mathfrak{m}$ such that z is not contained in $\mathfrak{a}$ and is a parameter for both M and M'. We show that $zH^0_{\mathfrak{m}}(M')=0$ contradicting the choice of z. Let $m' \in H^0_{\mathfrak{m}}(M')$. Then $m \in (xM :_M \mathfrak{m}^2)$. Since x is $M/H^0_{\mathfrak{m}}(M)$-regular, we have the following exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(M/(xM+H^0(M))) \longrightarrow H^1(M/H^0_{\mathfrak{m}}(M)) \xrightarrow{\ x\ } H^1(M/H^0_{\mathfrak{m}}(M))$$

and we have the isomorphisms

$$H^0_{\mathfrak{m}}(M/xM+H^0_{\mathfrak{m}}(M)) \cong H^1_{\mathfrak{m}}(M/H^0_{\mathfrak{m}}(M)) \cong H^1_{\mathfrak{m}}(M).$$

$zm \in xM + H^0_{\mathfrak{m}}(M)$, i.e., $zm = xn + t$ for some $n \in M$ and $t \in H^0_{\mathfrak{m}}(M)$. $xzm = x^2 n$ and $n \in (zM:x^2)$. Since $\{z,x^2\}$ is a s.s.o.p. for M,

we have $(0:x^2)_{M/zM} \subset H^0_{\mathfrak{m}}(M/zM) \subset (0:\mathfrak{m}^2)_{M/zM} \subset (0:x)_{M/zM}$.
Consequently, $xn=zu$ for some $u \in M$ and $t=zm - xn = z(m-u) \in zM$.

It follows that $t \in H^0_{\mathfrak{m}}(M) \cap zM = (0)$, for $H^0_{\mathfrak{m}}(M) = (0:z^i)_M$ for any $i \geq 1$. We get $zm = xn \in xM$ and $zm' = 0$, as was required.

Now let i be ≥ 1. Since $H^0_{\mathfrak{m}}(M) = (0:x^j)_M$ for all $j \geq 1$, we have an exact sequence $0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow M/xM \rightarrow M/(xM+H^0_{\mathfrak{m}}(M)) \longrightarrow 0$ and isomorphisms $H^i_{\mathfrak{m}}(M/xM) \cong H^i_{\mathfrak{m}}(\overline{M}/x\overline{M})$ for $i \geq 1$ with $\overline{M}=M/H^0_{\mathfrak{m}}(M)$. We may assume that depth(M) > 0 and hence x is M-regular.

Suppose $\mathfrak{a}:=\mathrm{ann}_A(H^i_{\mathfrak{m}}(M)) \subsetneq \mathfrak{m}$ and choose $\alpha \notin \mathfrak{a}$ so that α is a parameter for both M and M'=M/xM. Let $E^\bullet$ and $F^\bullet$ be the minimal injective resolutions of M and M', respectively. $e^\bullet$ and $f^\bullet$ be the differential maps. ${}^0E^\bullet := H^0_{\mathfrak{m}}(E^\bullet)$, ${}^0F^\bullet := H^0_{\mathfrak{m}}(F^\bullet)$,

${}^{o}e^{\bullet} := H^{0}_{\mathfrak{m}}(e^{\bullet})$ and ${}^{o}f^{\bullet} := H^{0}_{\mathfrak{m}}(f^{\bullet})$. Consider the exact sequence

$$0 \longrightarrow {}^{o}E^{\bullet} \xrightarrow{x} {}^{o}E^{\bullet} \longrightarrow {}^{o}F^{\bullet} \longrightarrow 0$$

of complexes induced from the exact sequence $0 \to M \xrightarrow{x} M \to M' \to 0$. Let $z \in {}^{o}E^{i}$ be arbitrary such that $z' = z \bmod x\,{}^{o}E^{i} \in \mathrm{Ker}({}^{o}f^{i})$. ${}^{o}e^{i}(z) = xw$ for some $w \in {}^{o}E^{i+1}$ with ${}^{o}e^{i+1}(w) = 0$. Since $\mathfrak{m}H^{i+1}_{\mathfrak{m}}(M) = 0$, there exists $u \in {}^{o}E^{i}$ such that

$$(\#) \quad \alpha w = {}^{o}e^{i}(u).$$

${}^{o}e^{i}(\alpha z) = \alpha\,{}^{o}e^{i}(z) = \alpha xw = {}^{o}e^{i}(xu)$ —— (##). Consequently $\alpha z - xu \in \ker {}^{o}e^{i}$. Since $\mathfrak{m}H^{i}_{\mathfrak{m}}(M) = 0$, we have $x^{2}u - \alpha zx \in \mathrm{Im}({}^{o}e^{i-1})$ and $x^{2}u \in \alpha\,{}^{o}E^{i} + \mathrm{Im}({}^{o}e^{i-1})$. Applying the same argument to the exact sequence $0 \to M \xrightarrow{\alpha} M \to M/\alpha M \to 0$, and the exact sequence of complexes $0 \to {}^{o}E^{\bullet} \xrightarrow{\alpha} {}^{o}E^{\bullet} \longrightarrow {}^{o}F^{\bullet} \longrightarrow 0$, we see that $x^{2}(u \bmod \alpha\,{}^{o}E^{i}) \in \mathrm{Im}({}^{o}f^{i-1})$ and (#) implies that $u \bmod \alpha\,{}^{o}E^{i} \in \mathrm{Ker}({}^{o}f^{i})$, i.e., $(u \bmod \alpha\,{}^{o}E^{i}) \in \mathrm{Ker}({}^{o}f^{i}) \cap (\mathrm{Im}({}^{o}f^{i-1}) : x^{2})$. Since $x \in \mathfrak{m}^{2}$, we have $H^{i}_{\mathfrak{m}}(M/\alpha M) \subset (0:\mathfrak{m}^{2})_{H^{i}_{\mathfrak{m}}(M/\alpha M)} \subset (0:x)_{H^{i}_{\mathfrak{m}}(M/\alpha M)} \subset (0:x^{2})_{H^{i}_{\mathfrak{m}}(M/\alpha M)} \subset H^{i}_{\mathfrak{m}}(M/\alpha M)$. Thus we have $x(u \bmod \alpha\,{}^{o}E^{i}) \in \mathrm{Im}({}^{o}f^{i-1})$, and there exists $v \in {}^{o}E^{i}$ such that $xu - \alpha v \in \mathrm{Im}({}^{o}e^{i-1})$ By (##) we have ${}^{o}e^{i}(\alpha v) = {}^{o}e^{i}(xu) = {}^{o}e^{i}(\alpha z)$, and

$$\alpha(v-z) \in \ker({}^{o}e^{i}) \cap \alpha({}^{o}E^{i}) \subset \mathrm{Im}({}^{o}e^{i-1}),$$

for $(0:\alpha^{j})_{H^{i}_{\mathfrak{m}}(M)} = H^{i}_{\mathfrak{m}}(M)$ for all $j \geq 1$.

We therefore have $\alpha z - xu = (\alpha z - \alpha v) + (\alpha v - xu) \in \mathrm{Im}({}^{o}e^{i-1})$. Namely $\alpha(z \bmod x\,{}^{o}E^{i}) \in \mathrm{Im}({}^{o}f^{i-1})$ and α kills $H^{i}_{\mathfrak{m}}(M/xM)$, contradicting the choice of α.

The proof of the latter half is just a slight modification of one of Vogel's Non-Zero-Divisor characterization of Buchsbaum modules [2].

As an easy consequence of the theorem, the following is proved.

COROLLARY(1.2)([3]). The following are equivalent.

(i) M is a quasi-Bbm module.

(ii) Any s.o.p. for M contained in $\mathfrak{m}^2$ forms a weak M-seq..

(iii) There exists a weak M-sequence of length $d = \dim(M)$ in $\mathfrak{m}^2$.

The next lemma connects the local cohomology to the Koszul homology and played an essential role in the proof of theorem (1.1).

LEMMA(1.3). Let M be a generalized Bbm module (i.e., for $i<d$ $L_A(H^i_{\mathfrak{m}}(M))<\infty$) and x be a parameter for M. Then

$$(0:x)_M \subseteq H^0_{\mathfrak{m}}(M).$$

Considering the long exact sequence of local cohomology we easily see that M/xM is also a generalized Bbm module for any parameter x for M. Since the Koszul complex is obtained by the successive construction of mapping cylinder, taking the lemma into account, we see that the Koszul homology $H_+(\underline{x};M)$ with respect to any s.s.o.p. $\underline{x}$ for a generalized Bbm module M has finite length and it is not hard to see the following.

PROPOSITION(1.4). Let M be a generalized Bbm module and $\underline{x} = \{x_1,\dots,x_d\}$ be a s.o.p. for M. Then

(i) $h^p(M/(x_1,\dots,x_r)M) \leqq \sum_{i=0}^{r}\binom{r}{i}h^{i+p}(M)$ for $r = 1,\dots,d-1$.

(ii) $h_p(x_1,\dots,x_r;M) \leqq \sum_{i=0}^{r-p}\binom{r}{p+i}h^i(M)$ for any $p \geqq 1$.

(iii) $L_A(M/(\underline{x})M) - e_0(\underline{x};M) = L_A((0:x_d)_{M/(x_1,\dots,x_{d-1})M})$.

If M is Bbm, then equality holds in (i). Moreover each

Koszul homology module is a vector space (and its length is expressed in terms of local cohomology). The fact conversly characterizes the Buchsbaum modules.

THEOREM(1.5)(Suzuki, Schenzel) The following are equivalent.
(i) M is a Bbm module.
(ii) $\mathfrak{m}H_1(\underline{x};M) = 0$ for any s.o.p. (resp. s.s.o.p.) $\underline{x}$ for M.
(iii) $\mathfrak{m}H_+(\underline{x};M)=0$ for any s.o.p. (resp. s.s.o.p.) $\underline{x}$ for M.

Schenzel's proof uses the dualizing complex, while that of the author's is elementary. Note also that $H_+(\underline{x};M)$ is the socle of $K_+(\underline{x};M)/B_+(\underline{x};M)$.

COROLLARY(1.6). Let $\underline{x}$ be a s.s.o.p. for a Bbm module M. Then

$$h_p(x_1,\dots,x_r;M) = \sum_{i=0}^{r-p}\binom{r}{p+i}h^i(M),$$

hence for any s.o.p. $\underline{x} = x_1,\dots,x_d$ for a Bbm module M, we have

$$L_A(M/(\underline{x})M)-e_0(\underline{x};M) = h^0(M/(x_1,\dots,x_{d-1})M)=\sum_{i=0}^{d-1}\binom{d-1}{i}h^i(M),$$

which is the invariant stated in Theorem (1.1).

In spite of the above facts, we must be careful when we consider the Koszul homology of weak sequences.
<u>REMARK</u>. (i) If $\underline{x}$ is contained in $\mathfrak{m}^2$, then $\mathfrak{m}H_1(\underline{x};M) = 0$ implies that $\underline{x}$ is a weak M-sequence.
(ii) If $\underline{x}$ is an unconditioned weak sequence (i.e., after any permutation it is still a weak sequence), it is not necessarily true that $H_1(\underline{x};M)$ is a vector space.

We close this section with the following which is quoted partly from the recent results by M. Steurich.

THEOREM.([5]). Let $x_1,\dots,x_n$ be a sequence of elements generating minimally the ideal $(x_1,\dots,x_n)$. Then the following are equivalent.

(i) $x_1,\dots,x_n$ is an unconditioned weak sequence.

(ii) $Z_1(x_i, i\in I)/(\mathfrak{M}(\underline{x})K_1(x_i, i\ I, i\neq i_o)\cap Z_1(x_i, i\in I)+B_1(x_i, i\in I))$ is a vector space for any $I\subset\{1,2,\dots,n\}$ and $i_o\in I$.

Note that if $\underline{x}$ is besides an unconditioned relatively $\mathfrak{M}$-regular sequence with respect to $\mathfrak{M}(\underline{x})$ in the sence of Fiorentini (10), the module in (ii) coincides with the usual Koszul homology.

§2. Bounds for the multiplicity of Buchsbaum modules.

Our next purpose is to prove the following

THEOREM. Let M be a Buchsbaum module of dimension d (≥ 2) and $\underline{x}$ be a system of parameters for M. Then we have

$$e_o(\underline{x};M) \geq \sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(M).$$

If A is a Buchsbaum ring of dimension d (≥ 2), then

$$e_o(\underline{x};A) \geq 1 + \sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(A).$$

Notation. $D^p(\ ) := \mathrm{Hom}_A(H^p_{\mathfrak{M}}(\ ),E_A(k))$. $D^d(M)$ is the so-called canonical module of M.

COROLLARY(2.0) If $e_o(\underline{x};M) < d-2$ for some s.o.p. $\underline{x}$ for M, then $h^i(M) = 0$ for $i = 2,\dots,d-1$ and $D^d(M)$ is a C.-M.module with $e_o(\underline{x};M) = L_A(D^d(M)/(\underline{x})D^d(M))$.

If $e_0(\underline{x};A) < d-1$, then $h^i_{\mathfrak{m}}(M)=0$ for $i=2,\dots,d-2$ and $h^{d-1} \leq 1$.

<u>LEMMA</u>(2.1). Let M be a generalized Bbm module of dimension $d \geq 2$. Then there exists an exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow D^d D^d(M) \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow 0$$

and isomorphisms $D^p D^d(M) \cong D^0(D^{d-p+1}(D))$ for $p=2,\dots,d-1$, and $D^0 D^d(M) = D^1 D^d(M) = 0$.

<u>LEMMA</u>(2.2). Let M be any finitely generated A-module and $\underline{x}$ be a s.o.p. for M. Then $e_0(\underline{x};D^d(M)) = e_0(\underline{x};M)$.

<u>PROPOSITION</u>(2.3)([6]). Let M be a Bbm module of positive depth and a be any M-regular element. Let U(aM) denote the unmixed component of the primary decomposition of aM in M:

$$U(aM) = \bigcap N(\mathfrak{p}) \quad \text{where } \mathfrak{p} \in \mathrm{ass}(M/aM) \text{ with } \dim(A/\mathfrak{p}) = d-1.$$

Then $U(aM) = (aM:b)_M = (aM:\mathfrak{m})_M$ for any parameter b for M/aM, and it is a Bbm module of dimension d. Furthermore we have an exact sequence

$$0 \longrightarrow M \xrightarrow{j} a^{-1}U(aM) \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow 0 \qquad \text{with } j(m) = a^{-1}(am).$$

<u>COROLLARY</u>(2.4). For a Bbm module M of positive depth, we have an isomorphism of A-modules; $a^{-1}U(aM) \cong D^d D^d(M)$.

Consequently $a^{-1}U(aM)$ is a Bbm module over A and does not depend on the element a.

LEMMA(2.5) Let M be a Bbm module of $\dim(M) \geq 2$. Then the number $v_A(D^d D^d(M))$ of minimal generators of $D^d D^d(M)$ is not less than $h^1(M)$.

If A is a Bbm ring of dimension $d \geq 2$, then

$v_A(D^dD^d(A)) = 1 + h^1(A).$

PROOF. The first inequality follows easily from (2.3). As to the second assertion, since we can choose $a \notin \mathfrak{m}^2$ and $U(aA) \subset \mathfrak{m}$, we can prove that $a^{-1}A \subsetneq (a^{-1}U(aA))$.

We are now ready to prove our main theorem.
Since $L_A(H^0_{\mathfrak{m}}(M)) < \infty$, we may assume that M is of positive depth. Let $\underline{x} = \{x_1,\ldots,x_d\}$ be any s.o.p. and $M' := M/(x_3,\ldots,x_d)M$. Since $\dim(M') = 2$, by (2.1) $D^2(M')$ is a C.-M.module. $e_0(\underline{x};M) = e_0(x_1,x_2;M') = e_0(x_1,x_2;D^2(M')) = L_A(D^2(M')/(x_1,x_2)D^2(M))$ and the last term is not less than the dimension of the socle, which coincides with $v_A(D^2D^2(M')$ and by (2.5) which is not less than $h^1(M')$ (resp. $= 1 + h^1(A)$ in the ring case.) On the other hand since M is a Bbm module we have $h^1(M') = \sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(M)$ by (1.4).

EXAMPLES. (i) Let $(R,\mathfrak{m},k)$ be a regular local ring of dim d. Then $M:=\mathfrak{m}$ is a d-dimensional Bbm module with depth = 1 and $h^1 = 1$ with $h^i = 0$ for $i \neq 1,d$. For the minimal generators $\underline{x}$ of $\mathfrak{m}$, we have $e_0(\underline{x};M) = e_0(\underline{x};R) = 1$. This is the case where the equality holds in the theorem.

(ii) Let $(R,\mathfrak{m},k)$ be as above. $A := R \ltimes \mathfrak{m} \supset \mathfrak{N} = \mathfrak{m} \ltimes \mathfrak{m}$, $\mathfrak{m} = (x_1,\ldots,x_d)$, and $y_i := (x_i,0)$ for $i=1,\ldots,d$. Then $h^1(A) = 1$ and $h^i(A) = 0$ for $i \neq 1,d$. On the other hand $L_A(A/(\underline{y})A) = d+1$, and we have $e_0(\underline{y};A) = d \geq 1+h^1(A)$. Equality holds if d=2.

(iii) In the ring case S.Goto proved

$$e_0(\underline{x};A) \geq 1 + \sum_{i=1}^{d-1}\binom{d-1}{i-1}h^i(A).$$

Let d=3 above, and L be the 2nd syzygy of k:

$0 \longrightarrow L \longrightarrow R^3 \longrightarrow R \longrightarrow k \longrightarrow 0$, and $A := R \ltimes L$. Then A is a Buchsbaum ring of dimension three with $h^2(A)=1$ and $h^i=0$ for $i \neq 2,3$. $e_0(\underline{y};A) = 3 = 1+2h^2(A)$. This is the example where the equality holds in Goto's inequality. He also asserts that if equality holds, A must be of maximal embedding dimension. Indeed the ring above has maximal embedding dimension.

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