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Rings With Good Formal Fibres

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§0. Introduction.

Let A, B be noetherian rings with ring-homomorphism ψ : $A \to B$. We know that the fibres of ψ hold a key to the ascent problems. Moreover we have a lot of beautiful theorems of this type, for example (2.1). But, in many cases, we are given only the conditions on few special fibres. So, in order to apply the ascent theory, it becomes necessary to obtain sufficient information on all fibres from the known data on special fibres.

Fix a local ring (A,m). Suppose for any noetherian ring B with ring-homomorphism $\psi : A \to B$, a good property on the fibre at the closed point implies the same one on any fibre. Then, putting $B = \hat{A}$ (= the m-adic completion of A) with canonical map $\rho : A \to \hat{A}$, A should have good formal fibres.

In this direction, Grothendieck constructed a general theory of rings with good formal fibres. There he left the following:

(0.1) Problem ([4, (7.5.4)]). Let (A,m), (B,n) be local rings with local homomorphism $\psi : A \to B$. Let P be a property of local rings. Suppose

(0.1.1) ψ is flat,

(0.1.2) $\bar{\psi} = \psi \otimes A/m$ is a P-homomorphism, and

(0.1.3) A is a P-ring.

Then is ψ also a P-homomorphism ? (for definitions, see §1.)

Concerning this question, André obtained an affirmative answer in the case where P = ((geometrically) regular). His result leads us to a positive answer for several P (cf. §3). But, for general P , we have only a partial result (cf. §5).

On the other hand, due to Krull, Zariski and Nagata, it is well-known that complete local rings themselves have many beautiful properties. So, not only in connection with the former problem, in order to study local rings, it is important to have a criteria for a local ring to have good formal fibres. Thanks to Nagata and Grothendieck, we know that if a local ring has good formal fibres then the rings of (essentially) finite type over it have also good formal fibres and that the formal fibres of any localization of a complete local ring are (geom.) regular.

With these facts in mind, Grothendieck expected that if a noetherian ring A has good formal fibres, i.e., the fibres of the canonical map $\rho_p : A_p \to (A_p)^\wedge$ (=the pA_p-adic completion of A_p) has a good property for any $p \in \mathrm{Spec}(A)$, then, for any ideal I of A , the I-adic completion A* of A also has good formal fibres. More generally, he asked the so-called "Lifting Problem":

(0.2) <u>Problem</u> ([4, (7.4.8)]). With A and I as above. Let P be a property of local rings. Suppose

(0.2.1) A is separated-complete in the I-adic topology,

(0.2.2) A/I is a P-ring.

Then is A also a P-ring ?

Rotthaus found a deep connection between the two Problems (0.1) and (0.2). She succeeded to prove Problem (0.2) for semi-local rings in the case where P = ((geom.) regular), showing that an affirmative answer to Problem (0.1) for some P implies an affirmative answer to Problem (0.2) for the same P. But, as she uses the "openness of P-loci" of complete (semi-) local rings in order to show the above implication, her method does not work well for non-local noetherian rings. Hence Problem (0.2) for general noetherian rings is still open even for P = ((geom.) regular) case (cf. (4.5), (4.6)).

§1. Notation and Terminology.

In this note, we use the standard notation and terminology as in Bourbaki [2], EGA [3,4], Matsumura [7] or Nagata[8]. A local ring means a noetherian ring with only one maximal ideal.

Let P be a property of local rings. We say " $P(A)$ is true" (or $P(A)$, for short), when a local ring A has property P. When k is a field and A is a local k-algebra, we use the notation $P(A,k)$ in place of $P(A)$, to emphasize the k-algebra structure of A. Similarly, if X is a locally noetherian scheme over a field k, we write $P(X,k)$ when $P(\mathcal{O}_{X,x},k)$ is true for any $x \in X$. With notation as above, we define:

(1.1) Definition. Let X, Y be locally noetherian schemes with a morphism $f : X \to Y$. Let P be a property of local rings. We call f a P-morphism if

(1.1.1) f is flat, and

(1.1.2) $P(f^{-1}(y),k(y))$ is true for any $y \in Y$.

The equivalent definition for rings is:

(1.2) <u>Definition.</u> Let A , B be noetherian rings with ring-homomorphism $\psi : A \to B$. Let P be a property of local rings. We call ψ a P-homomorphism if

(1.2.1) ψ is flat, and

(1.2.2) $P((B/pB)_P,k(p))$ is true for any $P \in \mathrm{Spec}(B)$, where $P \cap A = p$ and $k(p)$ denotes the residue field of A_p .

(1.3) <u>Definition.</u> Let A be a noetherian ring. Let P be a property of local rings. We call A a P-ring if, for any $p \in \mathrm{Spec}(A)$, the canonical map $\rho_p : A_p \to (A_p)^\wedge$ (= the pA_p-adic completion of A_p) is a P-homomorphism.

(1.4) <u>Definition.</u> In the case where P means (geom.) regular, normal, reduced-irreducible, or reduced, a P-ring is called a G-ring, a Z-ring, a C-ring, or an N-ring, respectively.

§2. <u>Conditions for P .</u>

Let P be a property of local rings. In many cases, P satisfies some of the following conditions (cf. (2.1)):

P_0 : For any field k , $P(k)$ is always true.

Let A , B be local rings with local homomorphism $\psi : A \to B$.

P_I : If ψ is a P-homomorphism, $P(A)$ implies $P(B)$.

P_{I*} : If ψ is regular, $P(A)$ implies $P(B)$.

P_{II} : If ψ is flat, $P(B)$ implies $P(A)$.

Let (A,m) be a local ring.

P_{III} : For any $p \in \mathrm{Spec}(A)$, P(A) implies $P(A_p)$.

P_{IV} : For a non-zero-divisor $t \in m$, P(A/tA) implies P(A) .

P_{IV*} : If A is catenary, for a non-zero-divisor $t \in m$, P(A/tA) implies P(A) .

Let k be a field and A a local k-algebra.

P_V : For any finite extension field k' of k and for any maximal ideal M of $A \otimes_k k'$. P(A,k) implies $P((A \otimes_k k')_M, k')$.

P_{VI} : If A is a complete local ring, $U_P(A) = \{ p \in \mathrm{Spec}(A) \mid A_p$ satisfies P } is open (in Zariski topology).

(2.1) <u>Table (Examples)</u>.

	I (I*)	II	III	IV (IV*)	V	VI
coprof ≤ n	× (○)	○	○	○	○	○
(S_n)	○	○	○	○	○	○
Cohen-Macaulay	◎	○	○	○	○	○
Gorenstein	◎	○	○	○	○	○
complete intersection	◎	○	○	○	○	○
(R_n)	○	○	○	(○)	*	○
n-normal $(=(S_{n+1})+(R_n))$	○	○	○	○	*	○
reduced	○	○	○	○	*	○
reduced-irreducible	Δ	○	○	○	Δ	○
normal	○	○	○	○	*	○
regular	◎	○	○	○	*	○

§3. Some Results for Problem (0.1).

The most remarkable answer to Problem (0.1) is:

(3.1) Theorem (André [1]). Let (A,m), (B,n) be local rings with local homomorphism $\psi : A \to B$. Suppose

(3.1.1) ψ is formally smooth, and

(3.1.2) A is quasi-excellent (= a G-ring).

Then ψ is regular.

(3.2) Remark. The condition (3.1.1) is equivalent to:

(3.1.1.1) ψ is flat, and

(3.1.1.2) $\bar{\psi} = \psi \otimes A/m$ is regular.

André's Theorem implies:

(3.3) Proposition (cf. [9,(2.4)]). With notation as above. Suppose

(3.3.1) ψ is flat, and

(3.3.2) $\bar{\psi}$ is n-normal, and

(3.3.3) A is an n-normal ring.

Then ψ is also n-normal.

In particular, putting $n = 1$ (or 0), suppose

(3.3.2.1) $\bar{\psi}$ is normal (or reduced, resp.), and

(3.3.3.1) A is a Z-ring (or an N-ring, resp.).

Then ψ is also normal (or reduced, resp.).

Similarly, one can show:

(3.4) Proposition. With notation as above. Suppose

(3.4.1) ψ is flat,

(3.4.2) $\bar{\psi}$ is reduced-irreducible, and

(3.4.3) A is a C-ring.

Then ψ is also reduced-irreducible.

As an application of Propositions (3.3) and (3.4), we have:

(3.5) Proposition (cf. [4, (7.5.7)], [8, (47.3)]). Let (A,m), (B,n), (W,ω) be local rings with local homomorphisms $\sigma : W \to A$ and $\tau : W \to B$. Suppose

(3.5.1) τ is flat,

(3.5.2) $B/\omega B$ is (geom.) normal (or reduced-irreducible, reduced) over W/ω,

(3.5.3) A/m is an extension of finite type over W/ω,

(3.5.4) A is a Z-ring (or a C-ring, an N-ring, resp.),

(3.5.5) A is normal (or unibranch reduced-irreducible, reduced, resp.).

Then the $m(A \otimes_W B)$-adic completion $(A \otimes_W B)^*$ of $A \otimes_W B$ is also normal (or (locally) reduced-irreducible reduced, resp.).
In particular, if B is complete, the complete tensor product $A \hat{\otimes}_W B$ of A and B over W is also normal (or (locally) reduced-irreducible, reduced, resp.).

§4. Some Results for Problem (0.2).

Thanks to Marot and Rotthaus, we have nice and fundamental results for Problem (0.2):

(4.1) Theorem (Marot [5]). Let A be a noetherian ring and

I an ideal of A . Suppose

(4.1.1) A is separated-complete in the I-adic topology,

(4.1.2) A/I is universally japanese (= pseudo-geometric).

Then A is also universally japanese.

In particular, if A is universally japanese then, for any ideal I of A , the I-adic completeion A* of A is also universally japanese,

(4.2) <u>Theorem</u> (Rotthaus [10]). Let A be a semi-local ring and I an ideal of A . Suppose

(4.2.1) A is separated-complete in the I-adic topology,

(4.2.2) A/I is quasi-excellent (= a G-ring).

Then A is also quasi-excellent.

In particular, if A is a semi-local quasi-excellent ring then, for any ideal I of A , the I-adic completion A* of A is also quasi-excellent.

As we remarked in Introduction, Proposition (3.3) implies:

(4.3) <u>Proposition</u>. With notation as above. Suppose

(4.3.1) A is separated-complete in the I-adic topology,

(4.3.2) A/I is an n-normal ring.

Then A is also an n-normal ring.

In particular, suppose

(4.3.2.1) A/I is a Z-ring (or an N-ring).

Then A is also a Z-ring (or an N-ring, resp.).

In the same way, we have:

(4.4) Proposition. With notation as above. Suppose

(4.4.1) A is separated-complete in the I-adic topology,

(4.4.2) A/I is a C-ring.

Then A is also a C-ring.

(4.5) Remark (cf. [9, (5.3)]). For general noetherian rings, the above propositions are not valid. We constructed a 1-dimensional noetherian domain A which has the following properties:

(4.5.1) A is a G-ring, but

(4.5.2) A[[X]] is not an N-ring.

So, in order to obtain a satisfactory answer to Problem (0.2) for non-local noetherian rings, we need to supply some missing global conditions. Here also Rotthaus got a good result:

(4.6) Theorem (Rotthaus [11]). Let A be a noetherian ring and I an ideal of A . Suppose

(4.6.1) $A \supset \mathbb{Q}$,

(4.6.2) A is universally catenary,

(4.6.3) A is separated-complete in the I-adic topology,

(4.6.4) A/I is excellent, and

(4.6.5) $\dim A < \infty$.

Then A is also excellent.

§5. Some Results in Characteristic 0 .

If a local ring contains the field of rational numbers $\mathbb{Q}$, Problem (0.1) is valid. In this section we give an elementary proof of this fact. But we don't know if we can apply the technique used in the proof of (5.1) to general cases, because the separability condition plays important roles in many places.

(5.1) Proposition (cf. Marot [6, Theorem 2.5]) Let (A,m), (B,n) be local rings with local homomorphism $\psi : A \to B$. Let P be a property of local rings. Suppose

(5.1.1) P satisfies the conditions P_I, P_{II}, P_{III}, P_{IV},

(5.1.2) $A \supset \mathbb{Q}$,

(5.1.3) ψ is flat,

(5.1.4) $\bar{\psi} = \psi \otimes A/m$ is a P-homomorphism,

(5.1.5) A is a P-ring.

Then ψ is also a P-homomorphism.

Marot showed that an affirmative answer to the resolution of singularities for excellent local rings implies a positive answer to Problem (0.1):

(5.2) Proposition (Marot ([6, Theorem 2.2], cf. [4, (7.9.8)], [12, Teorema 3.4]). With notation as above. Suppose

(5.2.1) P satisfies the conditions P_{III}, P_{IV}, P_V,

(5.2.2) ψ is flat,

(5.2.3) $\bar{\psi}$ is a P-homomorphism, and

(5.2.4) for any finite A-algebra C which is a domain, there exists a regular scheme Y and a proper birational morphism $g : Y \to \mathrm{Spec}(C)$.

Then ψ is also a P-homomorphism.

Proof of Proposition (5.1). First we note that, since P satisfies the condition P_I and A is a P-ring, we may replace A by its m-adic completion $\hat{A}$ and B by its mB-adic completion B^* (cf. [9, (2.4)]).

We prove the proposition by induction on $\dim A$. Suppose $\dim A \leq 1$. We may assume A is a 1-dimensional domain. Then, taking the integral closure $\bar{A}$ of A, we may also assume A is a discrete valuation ring. Hence, using the condition P_{IV}, we get the conclusion.

Let $\dim A = d$ (≥ 2). Suppose the assertion is valid for any quasi-excellent local ring of dimension less than d. Furthermore, as P satisfies the conditions P_I and P_{II}, we may assume B is henselian. By noetherian induction, we may also assume A is a domain and $\psi \otimes A/p$ is a P-homomorphism for any non-zero prime p. Hence it remains to show:

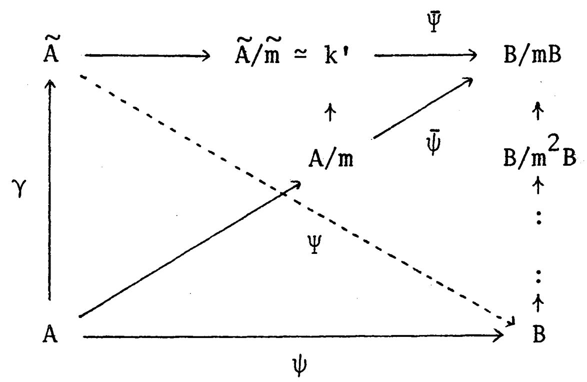
(5.1.6) If P is a prime ideal of B such that $P \cap A = (0)$ then B_P satisfies P.

Proof of (5.1.6). Take a non-zero element $a \in m$. Let Q be a minimal prime of $P + aB$. If $Q \cap A = q \subsetneq m$, by inductive assumption, $\psi_Q : A_q \to B_Q$ is a P-homomorphism. Thus we may assume $Q \cap A = m$. Also we may replace B by B_Q. Hence $\dim B/P = 1$.

Let k' be the residue field of B. Since $\operatorname{ch} k' = 0$ and B/mB is assumed to be henselian, by the structure theorem of complete local r ings ([8, (31.1)]), we see that B/mB has a coefficient field k' as an extension of the canonical injection $\bar{\psi}(A/m)$.

On the other hand, we can construct a complete local, flat A-algebra $(\tilde{A},\tilde{m})$ with canonical map $\gamma : A \to \tilde{A}$ such that $\tilde{m} = m\tilde{A}$ and $\tilde{A}/\tilde{m} \simeq k'$ ([3, 0_I(6.8.2)]). Then, as k' is separable over A/m, $\tilde{A}$ is formally smooth over A ([4, 0_{IV}(19.7.1)]). Hence, by André's Theorem(3.1), γ is regular.

Consider the following diagram:



Since $\tilde{A}$ is formally smooth over A, we have an A-algebra homomorphism $\Psi : \tilde{A} \to B$ such that $\Psi\circ\gamma = \psi$. Then

(5.1.3*) Ψ is flat (cf. [2, §5, n^{0}4, Proposition 3]), and

(5.1.4*) $\bar{\Psi} = \Psi \otimes \tilde{A}/\tilde{m}$ is a P-homomorphism.

Put $P \cap \tilde{A} = \tilde{p}$. Then B/P is a finite $\tilde{A}/\tilde{p}$-module, because $B/n \simeq \tilde{A}/\tilde{m}$, $P + aB$ is n-primary and $\tilde{A}$ is complete (cf. [3, 0_I(7.4.2)]). Hence $\dim \tilde{A}/\tilde{p} = 1$. This implies that $\Psi \otimes \tilde{A}/\tilde{p}$ is a P-homomorphism. Consequently, $B_P/\tilde{p}B_P$ satisfies P. Moreover, as $\tilde{p} \cap A = (0)$ and γ is regular, $\tilde{A}_{\tilde{p}}$ is regular. Therefore, using the condition P_{IV}, we see that B_P satisfies P. Thus the proof of (5.1.6) is finished and this completes the proof of (5.1).

Once we prove Proposition (5.1), we have (cf. Introduction):

(5.3) Proposition (cf. Marot [6, Theorem 3.2]). Let A be a semi-local ring, I an ideal of A. Let P be a property of local rings. Suppose

(5.3.1) P satisfies the conditions P_I, P_{II}, P_{III}, P_{IV*}, and P_{VI},

(5.3.2) $A \supset \mathbb{Q}$,

(5.3.3) A is separated-complete in the I-adic topology,

(5.3.4) A/I is a universally japanese P-ring.

Then A is also a (universally japanese) P-ring.

In particular, if $A \supset \mathbb{Q}$ and if A is a universally japanese semi-local P-ring, the I-adic completion A* of A is also a (universally japanese) P-ring.

(5.4) Remark. We don't know if the reducedness condition on formal fibres in (5.3.4) is indispensable.

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