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Menger-Decomposition of a Graph And Its Application to the Structural Analysis of a Large-Scale System of Equations

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Abstract

Graph representation of a large-scale system of non-linear equations provides an efficient way of testing the structural solvability, detecting the inconsistencies in modelling and decomposing the whole system into partially ordered subsystems.

In this paper, the M-decomposition is defined for a graph with specified "entrance" and "exit" vertices, in terms of the Menger-type linkings from the entrance to the exit. Some properties of the M-decomposition are shown; specifically it is noted that the M-decomposition agrees with the Dulmage-Mendelsohn decomposition of the associated bipartite graph.

The M-decomposition is useful for the structural analysis of a large-scale system of equations; the M-decomposition leads to the finest block-triangularization and the resulting subproblems are structurally solvable. Also pointed out is the fact that among the cycles on the representation graph, only those which are contained in an M-indecomposable component correspond to essential equations.

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1. Introduction

Graph-theoretic approach has turned out to be useful for the structural analysis of a large-scale physical/engineering system or a system of non-linear equations describing it. In particular graph-theoretic analysis of the representation graph leads to an efficient way of testing the structural consistency, detecting the inconsistent parts and decomposing the whole system into partially ordered subsystems [13], [14], [19], [20].

Following [13], we consider a system of non-linear equations in the standard form:

$$\begin{cases} y_i = f_i(x, u) & (i=1,\dots,M) \\ u_k = g_k(x, u) & (k=1,\dots,K), \end{cases} \tag{1.1}$$

where x_j $(j=1,\dots,N)$ and u_k $(k=1,\dots,K)$ are unknowns and y_i $(i=1,\dots,M)$ are parameters.

The <u>representation graph</u> $G(V,E)$ of (1.1) is a graph that represents the functional dependence among variables (i.e., unknowns and parameters). To be specific, $G(V,E)$ has the vertex set $V = X \cup U \cup Y$, where $X=\{x_1,\dots,x_N\}$, $U=\{u_1,\dots,u_K\}$ and $Y=\{y_1,\dots,y_M\}$. The functional dependence

$$y_i = f_i(x,u)$$

is expressed by a set of arcs coming into y_i from x_j and u_l which effectively appear on the right-hand side; similarly for

$$u_k = g_k(x,u).$$

The system (1.1) of equations is said to be <u>structurally solvable</u> if it has a structure which admits a unique solution for arbitrarily specified values of parameters y_i $(i=1,\dots,M)$. It has been shown in [13] that (1.1) is structurally solvable iff there exists a Menger-type (vertex-disjoint) complete linking from X to Y on the representation graph G. (See section 4.1 for detail.)

In this paper, we investigate the structure of the Menger-type linkings on the representation graph, with a view to obtaining the finest decomposition of the whole system into structurally solvable subsystems. In section 2, we first introduce a decomposition of a capacitated network, by exploiting the structure of minimum cuts. Then the M-decomposition is defined for a graph with "entrance" and "exit" vertices in terms of the Menger-type linkings between them. In section 3, it is noted that the M-decomposition of a graph agrees with the Dulmage-Mendelsohn decomposition of the associated bipartite graph and that it is a refinement of the L-decomposition introduced in [13].

In section 4, the M-decomposition is applied to the structural analysis of a large-scale system of non-linear equations. The finest block-triangularization is obtained through the M-decomposition of the representation graph. In particular, it is pointed out that, among the cycles on the representation graph, those which are completely contained in an M-indecomposable component correspond to essential equations. An extension of the M-decomposition is considered to deal with inconsistent parts. The M- and the L-decomposition are compared with each other with respect to the total amount of computation involved in solving the whole system of equations.

2. M-Decomposition of a Graph

2.1. Decomposition of a network by the minimum cuts

Before defining the M-decomposition for a graph, we prepare a decomposition of a network on the basis of the minimum cuts.

Consider a network $N(V,E,c)$, where V is the vertex set containing a source s and a sink t, E is the arc set, and c: $E \to \underline{R}^+$ is the function defining the capacities of arcs. Let

$$Z = \{S \subset V \mid s \in S,\ t \notin S\}.$$

For any S in Z, we refer to

$$C(S) = \{(u,v) \in E \mid u \in S,\ v \notin S\}$$

as the cut corresponding to S and define its capacity by

$$\rho(S) = \sum_{e \in C(S)} c(e).$$

As is easily verified, the function $\rho: Z \to \underline{R}$ is submodular:

$$\rho(S \cup T) + \rho(S \cap T) \leq \rho(S) + \rho(T) \quad (S,T \in Z). \tag{2.1}$$

We will utilize the technique for decomposing submodular systems [10], [11], [12], [16], as sketched below. The family L of the minimizers of ρ, i.e.,

$$L = \{S \in Z \mid \rho(S) = \min_{T \in Z} \rho(T)\}$$

constitutes a distributive lattice with respect to set inclusion. (In fact, for S and T in L, the condition (2.1) implies that $\rho(S \cup T) = \rho(S \cap T) = \rho(S) = \rho(T)$.) Let $V_0 (s \in V_0)$ and $V - V_\infty (t \in V_\infty)$ be the minimum and the maximum element of L. Then, the Jordan-Hölder theorem for modular lattices [2] may be interpreted as stating to the effect that L determines a unique partition $P = \{V_i\}_{i=1}^r$ of $V-(V_0 \cup V_\infty)$:

$$V - (V_0 \cup V_\infty) = \bigcup_{i=1}^{r} V_i \qquad (V_i \cap V_j = \emptyset,\ i \neq j), \tag{2.2}$$

as well as the partial order ($\geq$) on P [16]. (See the algorithm below.)

The partial order on P may be arbitrarily extended on $P \cup \{V_0, V_\infty\}$, but in connection with the M-decomposition, we extend it as follows:

$$\begin{aligned} V_0 \geq V_j &\iff \exists u(\neq s) \in V_0, \exists v \in V_j: (u,v) \in E \text{ or } (v,u) \in E \quad (1 \leq j \leq \infty) \\ V_j \geq V_\infty &\iff \exists w(\neq t) \in V_\infty, \exists v \in V_j: (w,v) \in E \text{ or } (v,w) \in E \quad (0 \leq j \leq r) \end{aligned} \tag{2.3}$$

(See Step 6 of the algorithm for finding the min-cut decomposition.)

In accordance with the partition (2.2) of the vertex set V, the arc set E is partitioned as

$$E = (\bigcup_{i=0}^{\infty} E_i) \cup (\bigcup_{i \neq j} E_{ij}),$$

$$E_i = E \cap (V_i x V_i) \qquad (i=0,1,\ldots,r,\infty)$$

$$E_{ij} = E \cap (V_i x V_j) \qquad (i \neq j; i,j=0,1,\ldots,r,\infty)$$

For each V_i $(1 \leq i \leq r)$, we define a network $N_i(V_i', E_i', c_i')$ with source s_i and sink t_i as follows:

$$V_i' = V_i \cup \{s_i, t_i\},$$

$$E_i' = E_i \cup \{(s_i,v) \mid \exists u, \exists V_j:(u,v) \in E_{ji}, V_j \gneq V_i\}$$
$$\cup \{(v,t_i) \mid \exists w, \exists V_j:(v,w) \in E_{ij}, V_i \gneq V_j\},$$

$$c_i'(e) = \begin{cases} c(e), & e \in E_i, \\ \sum_u c((u,v)), & e=(s_i,v) \\ \text{(summation taken over all u such that } (u,v) \in E_{ji}, & \\ \quad V_j \gneq V_i \text{ for some } V_j), & \\ \sum_w c((v,w)), & e=(v,t_i) \\ \text{(summation taken over all w such that } (v,w) \in E_{ij}, & \\ \quad V_i \gneq V_j \text{ for some } V_j). & \end{cases}$$

For V_0 (V_∞, resp.), we define N_0 (N_∞, resp.) in a similar manner by adding only the sink t_0 (the source s_∞, resp.).

The partition (2.2) of the vertex set V, or the decomposition of the network induced by it, will be referred to as the <u>min-cut decomposition</u> *. The resulting networks N_i, defined above, are called the <u>component networks</u>.

An algorithm for finding the min-cut decomposition (2.2), as well as the partial order, is given below [11], [17]. Throughout this paper, "v-*->w" means that there exists a directed path from v to w.

<u>Algorithm for min-cut decomposition of N(V,E,c)</u>

1. Find a maximum flow f from s to t on N(V,E,c) and fix it.
2. Define the "auxiliary graph" $G_f(V,\tilde{E})$ as follows: for u, v in V,

$$(u,v) \in \tilde{E} \iff [(u,v) \in E \text{ and } f((u,v)) < c((u,v))]$$
$$\text{or } [(v,u) \in E \text{ and } f((v,u)) > 0].$$

3. Let V_0 be the set of vertices v such that s-*->v on G_f.
4. Let V_∞ be the set of vertices v such that v-*->t on G_f.
5. Let $P=\{V_i\}_{i=1}^{r}$ be the collection of the strongly connected components of the graph obtained from G_f by deleting the vertices in $V_0 \cup V_\infty$.

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* The min-cut decomposition defined here agrees with the decomposition treated in [17]. In [17], however, the decomposition is derived from a dual point of view, that is, it is defined in a constructive manner with reference to a maximum flow, which complicates the characterization, such as uniqueness, of the decomposition. The partial order defined here is somewhat different from that in [17], though they agree with each other on P. The removal of the condition "u≠s, w≠t" in (2.3) makes our partial order identical with that in [17].

6. Define the partial order($\geqq$) on $P \cup \{V_0, V_\infty\}$ as follows: for $0 \leq i, j \leq \infty$,

$$V_i \geqq V_j \iff \text{there exists on } G_f \text{ a directed path from a vertex in } V_j \text{ to a vertex in } V_i \text{ which passes through neither } s \text{ nor } t.$$

As is easily seen from the definition of the min-cut decomposition and the max-flow min-cut theorem [7], [9], the following theorem holds [17].

Theorem 2.1. (i) The minimum cuts of N (w.r.t. (s,t)) are in one-to-one correspondence with the monotone bisections* (P^+, P^-) of the partially ordered set $P = \{V_i\}_{i=1}^{r}$ ([12], p.169, Theorem A.2; [16]). That is, minimum cuts correspond to those subsets S of V which are expressed as

$$S = V_0 \cup \Big(\bigcup_{V_i \in P^+} V_i \Big).$$

(ii) Each $N_i(V_i', E_i', c_i')$ with $1 \leq i \leq r$ has exactly two minimum cuts w.r.t. (s_i, t_i), namely one corresponding to $\{s_i\}$ and the other to $V_i' - \{t_i\} = \{s_i\} \cup V_i$. In particular, we have, for each i,

$$\sum_{u,v} c((u,v)) = \sum_{v,w} c((v,w)),$$

where, on the left-hand side, the summation is taken over all u, v such that $u \in V_j$, $v \in V_i$, $(u,v) \in E_{ji}$, $V_j \gneqq V_i$ for some V_j; on the right-hand side, over all v, w such that $v \in V_i$, $w \in V_j$, $(v,w) \in E_{ij}$, $V_i \gneqq V_j$ for some V_j.

(iii) N_0 has a unique minimum cut w.r.t. (s, t_0), i.e., the one

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* A bisection (P^+, P^-) of P is called a monotone bisection if, for any V_i in P^+ and V_j in P^-, the relation $V_j \geqq V_i$ never holds.

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corresponding to V_0. N_∞ has a unique minimum cut w.r.t. (s_∞,t), i.e., the one corresponding to $\{s_\infty\}$. The capacities of those cuts are equal to the capacity of the minimum cuts of N.

(iv) A maximum flow f on N can be expressed as the union of maximum flows on N_i. That is, for a collection of maximum flows f_i on N_i, a maximum flow f on N can be obtained by

$$f(e) = \begin{cases} f_i(e) & e \in E_i, \\ c(e) & e \in E_{ij},\ V_i \gneq V_j, \\ 0 & \text{otherwise.} \end{cases}$$

Conversely a maximum flow f on N determines maximum flows f_i on N_i as above; the arcs incident to s_i $(1 \leq i \leq \infty)$ or t_j $(0 \leq j \leq r)$ are to be saturated. ▯

2.2. Definition of the M-decomposition

Consider a graph G(V,E), with its vertex set V composed of three disjoint parts, i.e., $V = X \cup U \cup Y$, $X=\{x_1,\ldots,x_N\}$, $U=\{u_1,\ldots,u_K\}$ and $Y=\{y_1,\ldots,y_M\}$. Here it is assumed that there is no arc that comes into x_j in X or goes out of y_i in Y. We call X the entrance and Y the exit (including the case where $X = \emptyset$ or $Y = \emptyset$).

By a Menger-type linking from X to Y is meant a set of vertex-disjoint directed paths from a vertex in X to a vertex in Y. The size of a linking is defined to be the number of directed paths from X to Y contained in the linking. A linking of the maximum size is called a maximum linking and, in case $|X| = |Y|$, a linking whose size is equal to $|X|$ is called a complete linking. By a separator of (X,Y) is meant such a subset of the vertex set of G that has a common vertex with any directed path from a vertex in X to a vertex in Y. A separator of minimum cardinality is called a minimum separator.

We call here a vertex v of G an effective vertex if there exists on G a maximum linking that contains v. Those vertices which are not effective will be called ineffective vertices.

For a graph G(V,E) ($V=X \cup U \cup Y$) with the distinguished entrance X and exit Y, we define a network $N_G(\tilde{V},\tilde{E},c)$ with source s and sink t as follows, which will be called the network associated with G:

$$\tilde{V} = \{s,t\} \cup X_* \cup U_* \cup U^* \cup Y^*$$

$$X_*=\{x_*^1,\ldots,x_*^N\},\ U_*=\{u_*^1,\ldots,u_*^K\},$$

$$U^*=\{u_1^*,\ldots,u_K^*\},\ Y^*=\{y_1^*,\ldots,y_M^*\},$$

$$\tilde{E} = \tilde{E}_o \cup \tilde{E}_a$$

$$\tilde{E}_o=\{(v_*,w^*)\,|\,(v,w) \in E:\ v,w \in V\},$$

$$\tilde{E}_a=\{(s,x_*)\,|\,x \in X\} \cup \{(u^*,u_*)\,|\,u \in U\} \cup \{(y^*,t)\,|\,y \in Y\},$$

$$c(e) = \begin{cases} 1, & e \in \tilde{E}_a, \\ +\infty \text{ (sufficiently large)}, & e \in \tilde{E}_o. \end{cases}$$

As is well known [7], [9], there exists a one-to-one correspondence between Menger-type maximum linkings on G from X to Y and integral maximum flows on N_G from s to t which have no circulation (i.e., flow around a cycle). On the other hand, minimum separators of (X,Y) on G correspond to minimum cuts w.r.t. (s,t) on N_G.

Let

$$\tilde{V} = \tilde{V}_0 \cup \tilde{V}_\infty \cup \left(\bigcup_{i=1}^{r} \tilde{V}_i\right) \quad (s \in \tilde{V}_0,\ t \in \tilde{V}_\infty) \tag{2.4}$$

be the partition of $\tilde{V}$ determined by the min-cut decomposition of the associated network N_G.

Proposition 2.1. For $u \in U$, let $u^* \in \tilde{V}_i$ and $u_* \in \tilde{V}_j$ ($0 \le i,j \le \infty$). We have $\tilde{V}_i \geqq \tilde{V}_j$ or $\tilde{V}_j \geqq \tilde{V}_i$ according as u is effective or ineffective. ▯

Proof: If u is effective, then, by definition, there is a Menger-type maximum linking that contains u. For the maximum flow f on N_G corresponding to the linking, we have $f((u^*,u_*))=1$ and therefore an arc (u_*,u^*) exists on G_f, which implies that $\tilde{V}_i \geqq \tilde{V}_j$. Conversely, if u is ineffective, an arc (u^*,u_*) exists on G_f since $f((u^*,u_*))=0$ for any maximum flow f on N_G. Hence follows $\tilde{V}_j \geqq \tilde{V}_i$. Q.E.D.

For $\tilde{V}_i$ ($i=0,1,\dots,r,\infty$), set

$$m(\tilde{V}_i) = \{v \in V \mid v_* \in \tilde{V}_i\} \cup \{v \in V \mid v^* \in \tilde{V}_i\}.$$

The sets $m(\tilde{V}_i)$ ($i=1,\dots,r$) are not disjoint in general but are distinct with the following trivial exceptions.

Proposition 2.2. If $m(\tilde{V}_i) = m(\tilde{V}_j)$ for $1 \le i < j \le r$, then it is a singleton $m(\tilde{V}_i) = m(\tilde{V}_j) = \{u\}$ ($u \in U$). And u is an ineffective vertex. ▯

Proof: Suppose that $m(\tilde{V}_i) = m(\tilde{V}_j)$ and $\tilde{V}_i \ngeqq \tilde{V}_j$, and put $\tilde{V}_i=\{u_1^*,\dots,u_p^*,v_*^1,\dots,v_*^q\}$ and $\tilde{V}_j=\{u_*^1,\dots,u_*^p,v_1^*,\dots,v_q^*\}$. By Proposition 2.1, u_k ($1 \le k \le p$) are effective and v_k ($1 \le k \le q$) ineffective. Inspection into the arcs on the auxiliary graph reveals that only the case with p=0 and q=1 is possible. Q.E.D.

Let $\{V_j\}_{j=1}^R$ ($R \le r$) be the family of the distinct sets in $\{m(\tilde{V}_i)\}_{i=1}^r$. Also set $V_0=m(\tilde{V}_0)$ and $V_\infty=m(\tilde{V}_\infty)$. (We may have $V_0=\emptyset$ or $V_\infty=\emptyset$, whereas $V_j \ne \emptyset$ for $1 \le j \le R$.) Obviously, we have

$$V = V_0 \cup V_\infty \cup \left(\bigcup_{j=1}^{R} V_j \right). \tag{2.5}$$

The partial order ($\geqq$) in the min-cut decomposition (2.4) of N_G induces a partial order ($\geqq$) on $\{V_j\}_{j=0}^{\infty}$ by

$$V_j \geqq V_{j'} \quad (0 \leq j, j' \leq \infty) \iff V_j = m(\tilde{V}_i), V_{j'} = m(\tilde{V}_{i'}) \text{ and } \tilde{V}_i \geqq \tilde{V}_{i'}.$$

The decomposition* (2.5) of V, along with the partial order on it, will be referred to as the <u>M-decomposition</u> of G w.r.t. (X,Y). We call each V_j an <u>(M-)indecomposable component</u>, $\{V_j\}_{j=1}^{R}$ the <u>consistent part</u>, V_0 the <u>maximal inconsistent part</u>, V_∞ the <u>minimal inconsistent part</u> **. Those vertices which belong to two indecomposable components are called <u>connectors</u>.

For the associated network N_G, we define the M-decomposition

$$\{\bar{V}_0, \bar{V}_\infty\} \cup \{\bar{V}_j\}_{j=1}^{R} \qquad (2.6)$$

with the partial order ($\geqq$) in a similar manner by setting $\bar{V}_0 = \tilde{V}_0 - \{s\}$, $\bar{V}_\infty = \tilde{V}_\infty - \{t\}$ and merging the trivial components of the min-cut decomposition (2.4) such that $m(\tilde{V}_i) = m(\tilde{V}_{i'})$ $(1 \leq i < i' \leq r)$ (mentioned in Proposition 2.2) into single components. There is a natural one-to-one correspondence between the indecomposable components of the M-decomposition (2.6) of N_G and those of the M-decomposition (2.5) of G.

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* It should be clear that V_j's in (2.5) are not necessarily disjoint.

** The inconsistent parts could be decomposed further in an obvious manner if they are decomposed into several connected components after the consistent part is deleted from G. Then the partial order should be modified appropriately to represent the hierarchical structure more faithfully.

2.3. Properties of the M-decomposition

A component V_j ($1 \le j \le R$) in the consistent part of the M-decomposition will be called an <u>effective component</u> if it contains an effective vertex, and an <u>ineffective component</u> if it is composed of ineffective vertices only.

For each component V_j ($0 \le j \le \infty$) of the M-decomposition (2.5) of G, we define its entrance X_j ($\subset V_j$) and exit Y_j ($\subset V_j$) by

$$\begin{aligned} X_j &= (X \cap V_j) \cup \{u \in V_j \mid \exists V_i : u \in V_i \cap V_j,\ V_i \not\geq V_j\}, \\ Y_j &= (Y \cap V_j) \cup \{u \in V_j \mid \exists V_i : u \in V_i \cap V_j,\ V_j \not\geq V_i\}. \end{aligned} \tag{2.7}$$

Note that $X_j = Y_j = \emptyset$ for an ineffective component and that the connectors belonging to V_j are contained in $X_j \cup Y_j$. Let G_j be the graph obtained from the vertex-induced subgraph of G on V_j by deleting all the arcs coming into X_j or going out of Y_j.

With the above definitions, we have the following theorem.

<u>Theorem 2.2</u>. (i) The minimum separators of (X,Y) on G are in one-to-one correspondence with the monotone bisections (P^+, P^-) of the partially ordered set $P = \{V_j\}_{j=1}^{R}$. That is, a subset of V is a minimum separator iff it is expressed as

$$(V_0 \cup X \cup (\bigcup_{V_j \in P^+} V_j)) \cap (V_\infty \cup Y \cup (\bigcup_{V_j \in P^-} V_j)).$$

(ii) An effective component V_j is by itself the only indecomposable component in the M-decomposition of G_j w.r.t. (X_j, Y_j), i.e., it is indecomposable in this sense. (Hence, there is no inconsistent part there.)

(iii) For an effective component V_j, there exists a complete linking on G_j from X_j to Y_j; in particular $|X_j| = |Y_j| > 0$.

(iv) If $V_0 \ne \emptyset$, then $|X_0| > |Y_0|$ and the size of the maximum linking from

X_0 to Y_0 on G_0 is equal to $|Y_0|$. If $V_\infty \neq \emptyset$, then $|X_\infty| < |Y_\infty|$ and the size of the maximum linking from X_∞ to Y_∞ on G_∞ is equal to $|X_\infty|$.

(v) A maximum linking on G can be expressed as the union of complete linkings on the effective components and the maximum linkings on the inconsistent parts.

(vi) An ineffective component may be deleted without affecting the maximum linkings.

(vii) The existence of a complete linking on G is equivalent to $V_0=V_\infty=\emptyset$.

(viii) Connectors are the effective vertices which are contained in every maximum linking. ▯

Proof: Immediate from the properties of the min-cut decomposition of the associated network N_G given in Theorem 2.1. Q.E.D.

A comment would be in order as to the computational complexity of the M-decomposition. By virtue of the special form of the capacity of the associated network N_G, a maximum flow f on N_G can be found in $O(|E|\sqrt{|V|})$ time [6]. The strongly connected components of G_f are found in $O(|E|)$ time [1]. Hence the total amount of computation for determining the M-decomposition is $O(|E|\sqrt{|V|})$.

2.4. An example of the M-decomposition

Consider the graph G shown in Fig.1, where $X=\{x_1, x_2\}$, $Y=\{y_1, y_2, y_3\}$ and $U=\{u_1,\ldots,u_{11}\}$. Take the maximum linking $\{x_1\text{-->}u_1\text{-->}u_2\text{-->}y_1, x_2\text{-->}u_5\text{-->}u_6\text{-->}u_7\text{-->}y_2\}$ on G. The associated network N_G, together with the maximum flow f corresponding to the linking above, is shown in Fig.2. The min-cut decomposition of N_G, which is found by means of the auxiliary graph G_f in Fig.3, yields the M-decomposition of G and N_G (Figs.1, 2) as well as the partial order depicted in Fig.4. In this example, Proposition 2.2 applies to the part $m(\tilde{V}_8)=m(\tilde{V}_9)=\{u_{10}\}$; V_1 through V_5 are effective components, while V_6 through V_8 ineffective; u_1, u_2, u_5, u_6 and u_7 are connectors.

3. Relation between the M-Decomposition and Other Graph-Theoretic Decompositions

3.1. DM-decomposition of the associated bipartite graph

We begin with the following Theorem 3.1 which is an observation from the network-flow theoretical viewpoint.

For a graph $G(V,E)$ ($V=X \cup U \cup Y$) with disjoint entrance X and exit Y, the associated bipartite graph $B_G(V_*,V^*;\tilde{E})$ is defined as follows [13]:

$$V_* = X_* \cup U_*, \quad V^* = Y^* \cup U^*,$$

$$(v_*,w^*) \in \tilde{E} \iff (v,w) \in E \text{ or } v=w \in U.$$

Note that no arc exists on G that comes into X or goes out of Y. There is a natural one-to-one correspondence between the vertices of the associated bipartite graph B_G and those of the associated network N_G which are distinct from s and t.

Moreover, a maximum matching on B_G can be determined in accordance with a maximum linking on G: first, each directed path $x \to u_1 \to u_2 \to \cdots \to u_m \to y$ ($x \in X$, $u_l \in U$, $y \in Y$) in the linking on G determines the matching (x_*,u_1^*), (u_*^1,u_2^*), ..., (u_*^{m-1},u_m^*), (u_*^m,y^*) on B_G; next, each vertex u in U not contained in the linking induces the matching (u_*,u^*) on B_G; the union of those two kinds of matchings is a maximum matching on B_G. Note here that (u_*,u^*) is out of the matching if u (in U) is contained in the

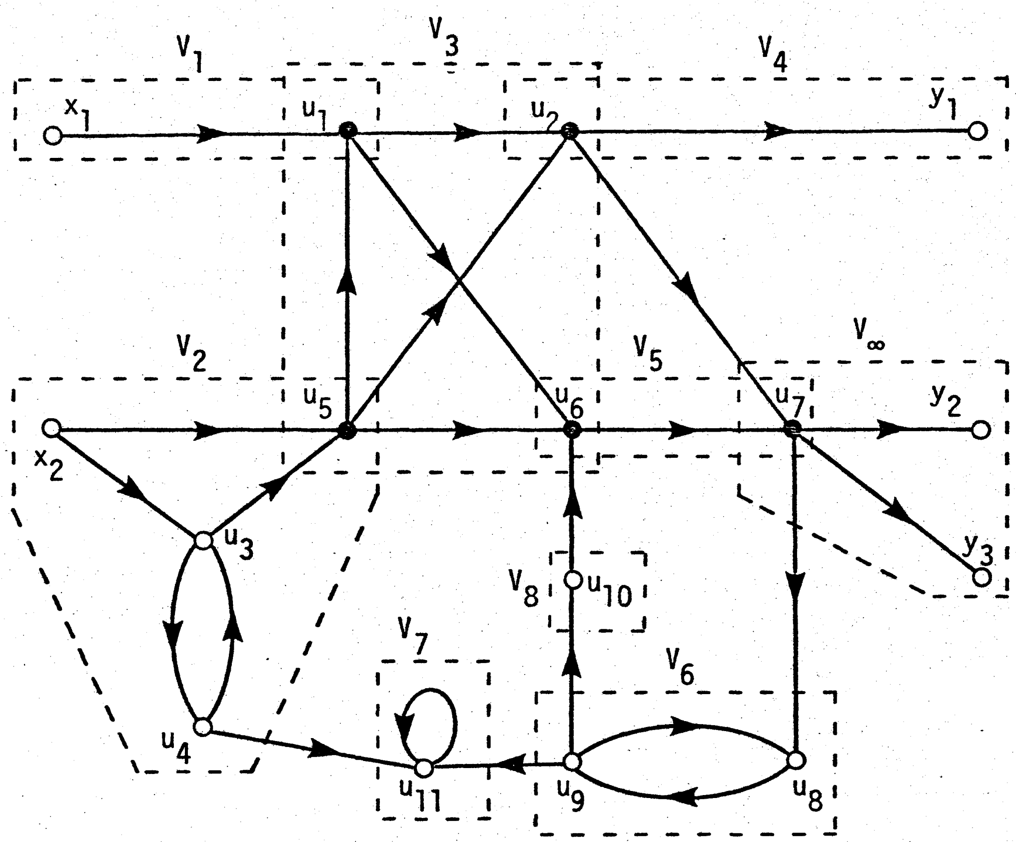

Fig.1. A graph G and its M-decomposition

⌐ ⌐ : M-decomposition (2.5) ($V_0 = \emptyset$)

V_1, V_2, V_3, V_4, V_5: Effective components

V_6, V_7, V_8: Ineffective components

●: Connectors

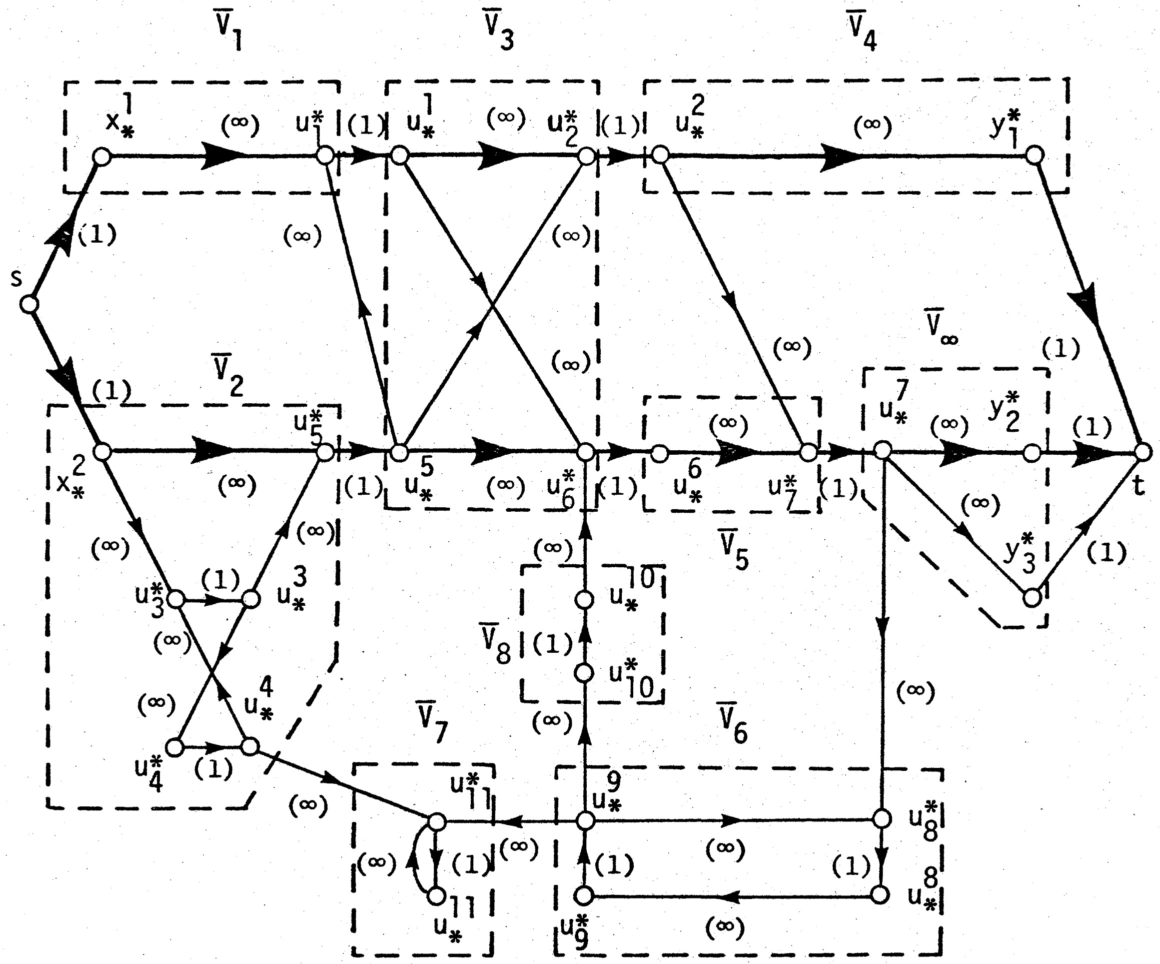

Fig.2. The associated network N_G of G and its M-decomposition

( ): Capacity

—: A maximum flow f

[dashed box] : M-decomposition (2.6) ($\overline{V}_0 = \emptyset$)

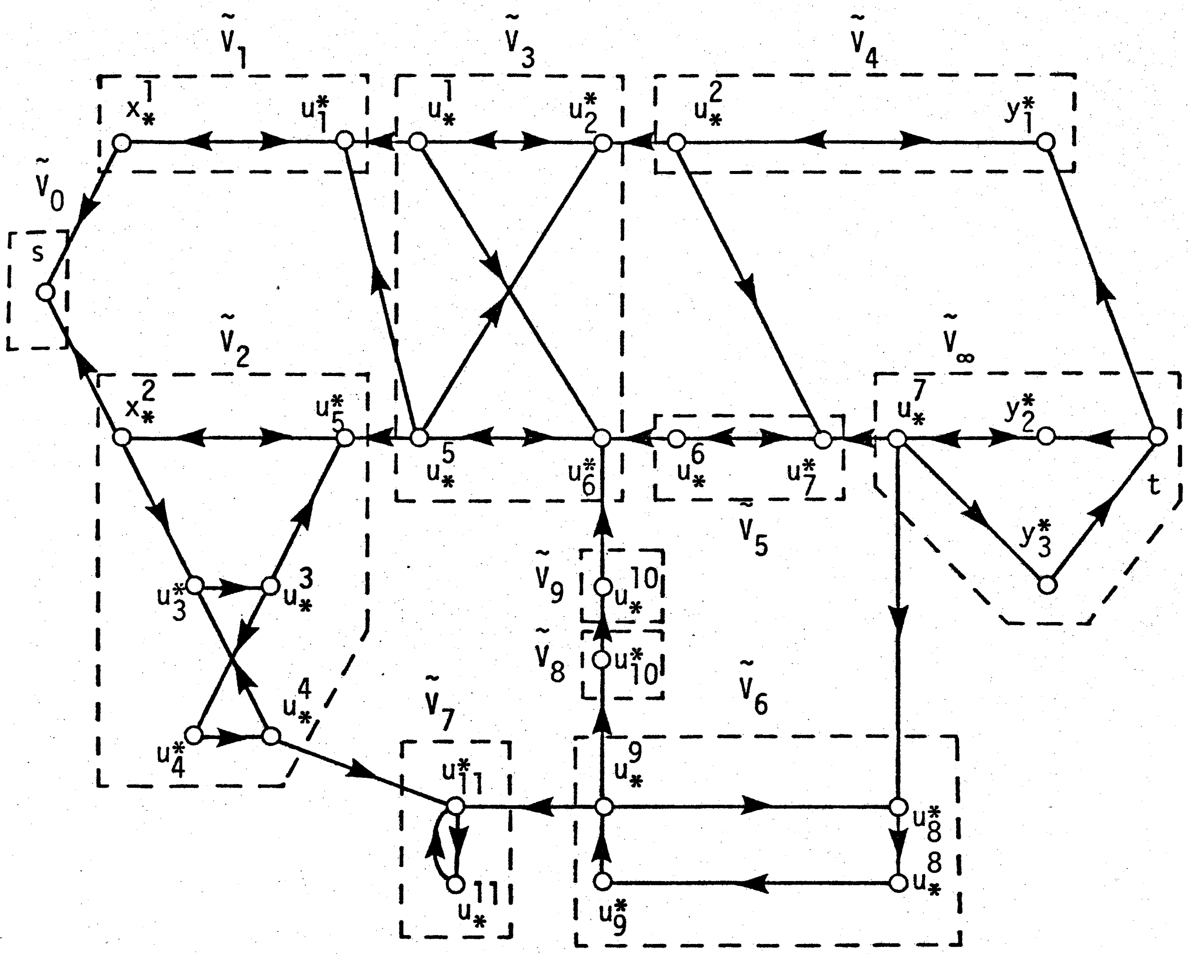

Fig.3. The auxiliary graph G_f corresponding to the maximum flow f on associated network N_G

[ ] : Min-cut decomposition

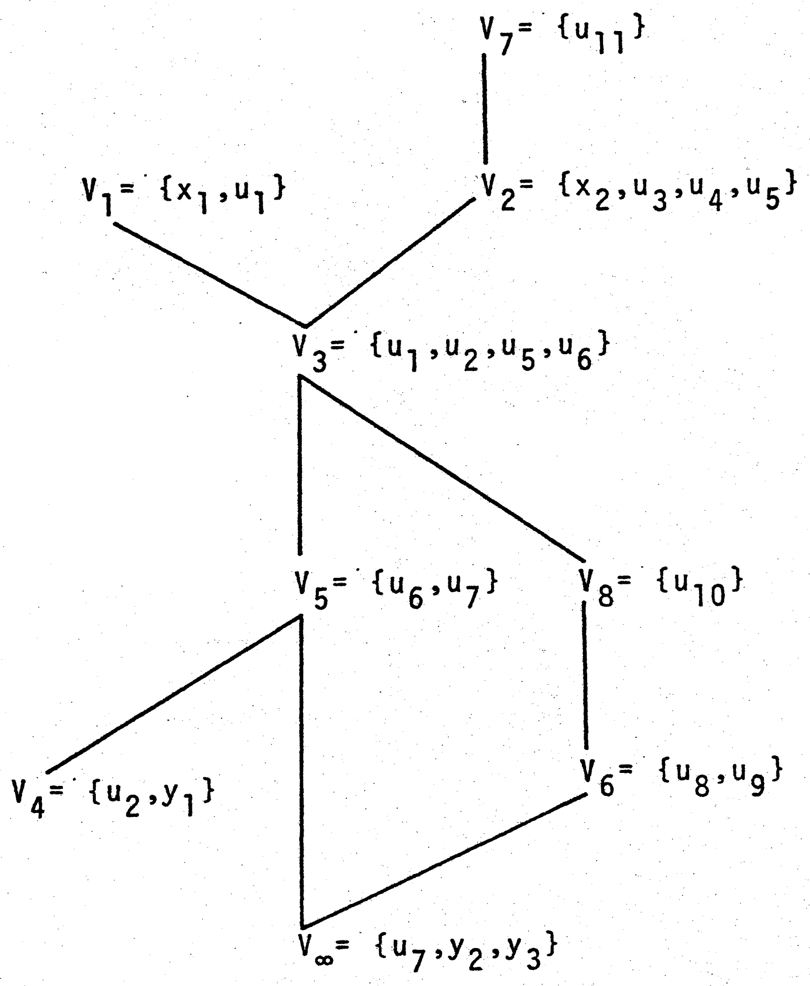

Fig.4. The Hasse diagram representing the partial order of the M-decomposition of G.
($V_0 = \emptyset$ has no relation with others.)

corresponding maximum linking.

By the DM-decomposition we mean the decomposition of a general bipartite graph $B(W_*,W^*;E)$ due to Dulmage and Mendelsohn [3], [4], [5], [12]. It can be found by the following procedure ([5]; [12], p.209).

Algorithm for the DM-decomposition of $B(W_*,W^*;E)$

1. Find a maximum matching M on $B(W_*,W^*;E)$ and fix it.
2. Define the auxiliary graph $G_M(W_* \cup W^*, \tilde{E})$ as follows:

$$(v,w) \in \tilde{E} \iff [(v,w) \in E,\ v \in W_*,\ w \in W^*]$$
$$\text{or } [(w,v) \in M,\ w \in W_*,\ v \in W^*].$$

3. Let W_0 be the set of vertices v such that $w \overset{*}{\to} v$ on G_M for some w in W_* which is not covered by M.
4. Let W_∞ be the set of vertices v such that $v \overset{*}{\to} w$ on G_M for some w in W^* which is not covered by M.
5. Let W_i (i=1,...,p) be the strongly connected components of the graph obtained from G_M by deleting the vertices of $W_0 \cup W_\infty$ and the arcs incident thereto.
6. Define the partial order $\geqq$ on $\{W_0, W_\infty\} \cup \{W_i\}_{i=1}^{p}$ as follows*: for $0 \leq i,j \leq \infty$,

$$W_i \geqq W_j \iff w_j \overset{*}{\to} w_i \text{ on } G_M \text{ for some } w_i \text{ in } W_i \text{ and } w_j \text{ in } W_j.$$

---

* The partial order concerning W_0 or W_∞ defined in [12] is slightly different, i.e., $W_0 \geqq W_i \geqq W_\infty$ for any i ($1 \leq i \leq p$) according to the definition in [12].

We will call W_0 (W_∞, resp.) the maximal (minimal, resp.) inconsistent part and $\{W_i\}_{i=1}^{P}$ the consistent part. (In [5] they are called the vertical (horizontal, resp.) tail and the core.)

The following theorem elucidates the relation between the M-decomposition of a graph and the DM-decomposition of its associated bipartite graph.

Theorem 3.1. For a graph G with entrance and exit specified, the M-decomposition (2.6): $\{\bar{V}_0, \bar{V}_\infty\} \cup \{\bar{V}_j\}_{j=1}^{R}$ of the associated network N_G agrees, inclusive of the partial order, with the DM-decomposition $\{W_0, W_\infty\} \cup \{W_j\}_{j=1}^{P}$ of the associated bipartite graph B_G. ▯

Proof: As usual, we transform the maximum matching problem on B_G to a max-flow problem by adding to B_G the source s and the sink t, and connecting s and t with the vertices of W_* and of W^*, respectively. Consider the auxiliary network N_B which corresponds to the matching $\{(u_*^k, u_k^*)\}_{k=1}^{K}$ on B_G, where $K=|U|$. The assertion of the theorem follows from the fact that N_G is identical with the network obtained from N_B by deleting the arcs (u_*^k, s) and (t, u_k^*) $(k=1,\ldots,K)$. Q.E.D.

The associated bipartite graph B_G of the graph G (in Fig.1) is given in Fig.5. Fig.6 illustrates that the DM-decomposition of B_G agrees with the M-decomposition of N_G in Fig.2.

3.2. L-decomposition

Consider a graph G(V,E) ($V=X \cup U \cup Y$) with disjoint entrance X and exit Y such that there exists a Menger-type complete linking from X to Y. The

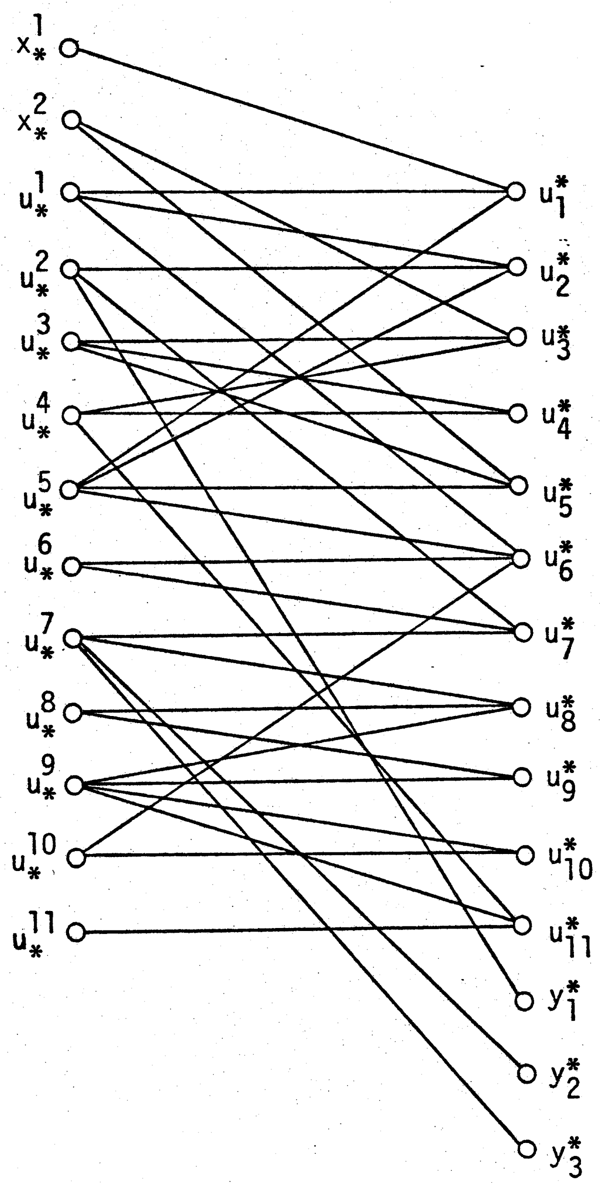

Fig.5 The associated bipartite graph B_G of G

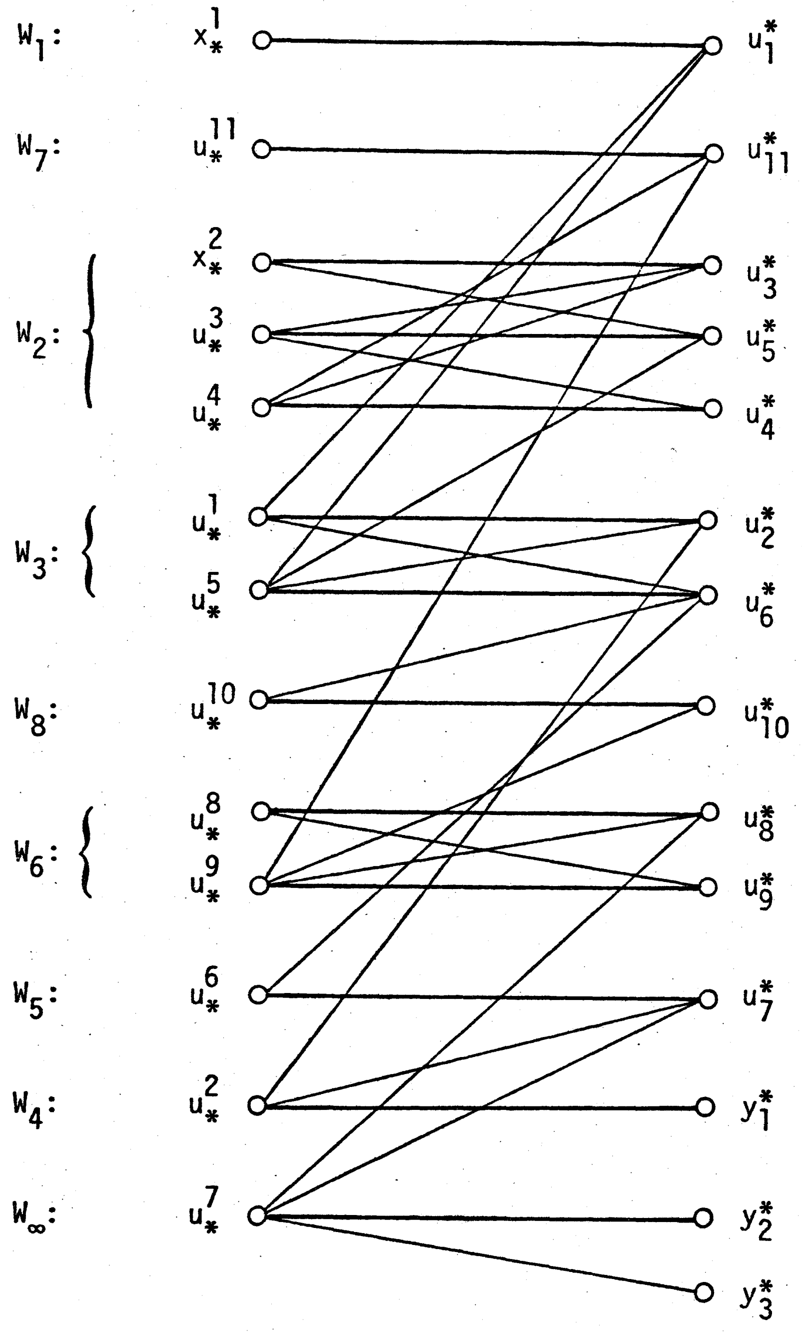

Fig.6. The DM-decomposition of the associated bipartite graph B_G

—— : The maximum matching corresponding to the maximum flow f on N_G

($W_0 = \emptyset$)

<u>L-decomposition</u> of G with respect to (X,Y) is defined as follows [13]. First fix a Menger-type complete linking from X to Y on G. Then construct a graph G' from G by identifying each pair of vertices x, y ($x \in X$, $y \in Y$) which are linked by the linking. The strongly connected components of the graph G' thus constructed determine a partition of the vertices of G in a natural manner. This partition of the vertices of G, together with the partial order induced from that among the strongly connected components of the graph G', is called the L-decomposition of G with respect to (X,Y). It is known that the L-decomposition is uniquely determined independently of the choice of the complete linking ([13], Theorem 3.1). An alternative characterization is given below in connection with the DM-decomposition of the associated bipartite graph.

<u>Proposition 3.1</u>. Suppose that there exists a Menger-type complete linking from X to Y on G. Let $\tilde{B}_G$ be the bipartite graph obtained from the associated bipartite graph B_G by adding an edge (x_*,y^*) to B_G for each pair of vertices x (in X) and y (in Y) such that $x \overset{*}{\rightarrow} y$ on G. Then for each u in U, u_* and u^* belong to the same component in the DM-decomposition of $\tilde{B}_G$, and the DM-decomposition of $\tilde{B}_G$ agrees with the L-decomposition of G with the natural correspondence between the vertices of $\tilde{B}_G$ and those of G. ▯

<u>Proof:</u> Let ($x_i \in X$, $y_i \in Y$) ($i=1,\dots,N$) be the linked pairs in a Menger-type complete linking on G. Then there exists on $\tilde{B}_G$ the complete matching $\{(x_*^i,y_i^*)\}_{i=1}^N \cup \{(u_*^k,u_k^*)\}_{k=1}^K$. This implies that (u_*,u^*) ($u \in U$) is an effective edge [12] (called an admissible edge in [3], [4], [5]), and therefore u_* and u^* belong to the same component in the DM-decomposition of $\tilde{B}_G$.

By definition, the L-decomposition of G is identical with the decomposition of G' into strongly connected components. On the other hand, as is easily seen, the DM-decomposition of $\tilde{B}_G$ agrees with the DM-decomposition of the associated bipartite graph of G'. The relation ([5]; [12], p.166, Theorem 6.6) between the decomposition of a graph into strongly connected components and the DM-decomposition of its associated bipartite graph establishes the proposition. Q.E.D.

The following theorem relates the L-decomposition to the M-decomposition.

Theorem 3.2. For a graph G with a Menger-type complete linking from X to Y, the M-decomposition is a refinement of the L-decomposition, inclusive of the partial order among the components. The L-decomposition is an order-homomorphic image of the M-decomposition; two M-indecomposable components with a connector in common, as well as those M-components lying between the two components with respect to the partial order, are to be merged into one to yield the L-decomposition. ▯

Proof: The first half is evident from Theorem 3.1 and Proposition 3.1. Let G_f be the auxiliary graph of N_G corresponding to a fixed Menger-type complete linking. Merging the M-components with common connectors as well as the intermediate M-components is equivalent to decomposing G_f into strongly connected components after identifying u_* and u^* for all u in U. This, in turn, is equivalent to decomposing G_f into strongly connected components after identifying those vertices which lie on each directed path from X to Y contained in that complete linking. The final decomposition is nothing but the L-decomposition. Q.E.D.

We will consider the case where there is a complete linking from X to Y on G. As far as the decomposition of $X \cup Y$ is concerned, the L-decomposition ignores the internal structure of G, in the sense to be made precise below.

Consider a bipartite graph $B_G^*(X,Y;E^*)$ which has an edge (x,y) iff $x \overset{*}{\rightarrow} y$ on G. This is called in [18] the underlying bipartite graph of the gammoid G(X,Y) in the context of linking systems.

Proposition 3.2. Suppose there exists a Menger-type complete linking from X to Y on G. The decomposition of $X \cup Y$ induced by the L-decomposition of G agrees with the DM-decomposition of B_G^*. ▯

Proof: Let $(x_i,y_i) \in X \times Y$ (i=1,...,N) be the linked pairs in a fixed complete linking from X to Y. Then B_G^* has the complete matching $\{(x_i,y_i)\}_{i=1}^N$. Consider a graph G* with vertex set X which has an arc (x_i,x_j) iff $x_i \overset{*}{\rightarrow} y_j$ on G and $i \neq j$. Evidently B_G^* is the associated bipartite graph of G*. The rest of the proof is the same as that of Proposition 3.1. Q.E.D.

Finally it may be remarked that the L-decomposition restricted to $X \cup Y$ agrees with the decomposition defined in [15] for a linking system [18].

4. Structural Analysis of a Large-Scale System of Equations

4.1. M-decomposition of the representation graph

In this section the M-decomposition is applied to the structural analysis of a system of equations. The following result is known [13] concerning whether or not the system (1.1) of equations has a structure which admits a unique solution for arbitrarily specified values of parameters y_i (i=1,...,M).

Theorem 4.1 ([13], Theorem 2.3). A system of equations in the standard form is structurally solvable* iff there exists on the representation graph a Menger-type complete linking from X to Y. ▯

Let G(V,E) ($V=X \cup U \cup Y$) be the representation graph of the system (1.1) of equations, where X and Y are the entrance and the exit of G, respectively. It is shown in [13] that the L-decomposition of the representation graph leads to a block-triangularization of (1.1), i.e., a decomposition of (1.1) into hierarchical subproblems. The M-decomposition brings about another block-triangularization which, by Theorem 3.2, is in general finer than that by the L-decomposition. Each M-indecomposable component V_j corresponds to a subproblem with parameters Y_j (defined in (2.7)) and unknowns V_j-Y_j, where G_j defined in section 2.3 is the

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* We consider the "general" case where the partial derivatives of f_i (i=1,...,M) and g_k (k=1,...,K) with respect to x_j (j=1,...,N) and u_l (l=1,...,K) can be regarded as elements of some extension field $\underline{F}$ of the rational number field $\underline{Q}$ and they are algebraically independent over $\underline{Q}$. The system (1.1) of equations is said to be structurally solvable if, under the generality assumption above, u_k (k=1,...,K) can be eliminated in (1.1) and the Jacobian of the resulting system of equations with unknowns x_j (j=1,...,N) is not equal to zero as an element of $\underline{F}$.

representation graph of that subproblem. The unknowns $V_j - Y_j$ will be called the inherent unknowns of this subproblem.

The theorem below follows from the condition for the structural solvability (Theorem 4.1) combined with the properties of the M-decomposition (Theorem 2.2).

Theorem 4.2. (i) A subproblem corresponding to an M-indecomposable component V_j ($1 \le j \le R$) in the consistent part is structurally solvable and cannot be further decomposed with the structural solvability maintained. It has a structure that admits a unique solution if the values of all the variables belonging to some V_i such that $V_j \gneq V_i$ are determined. (This statement holds true even when V_0 or V_∞ is non-empty.)

(ii) The subproblems corresponding to the inconsistent parts V_0, V_∞, if they exist, are not structurally solvable. The problem corresponding to V_0 is underdetermined i.e., has more unknowns than equations, and that to V_∞ is overdetermined, i.e., has fewer unknowns than equations.

(iii) (1.1) is structurally solvable iff $V_0 = V_\infty = \emptyset$. ▯

Let us consider an example:

$$\begin{cases} y_1 = f_1(u_2) \\ y_2 = f_2(u_2, u_7) \\ u_1 = g_1(x_1, u_2, u_3, u_7) \\ u_2 = g_2(u_1, u_5) \\ u_3 = g_3(x_2, u_4) \\ u_4 = g_4(x_2, u_3) \\ u_5 = g_5(u_3, u_4) \\ u_6 = g_6(u_1, u_5, u_{10}) \\ u_7 = g_7(u_6) \\ u_8 = g_8(u_7, u_9) \\ u_9 = g_9(u_8) \\ u_{10} = g_{10}(u_9) \\ u_{11} = g_{11}(u_4, u_9, u_{11}) \end{cases} \tag{4.1}$$

The representation graph G, as well as its M-decomposition, is shown in Fig.7. The system (1.1) of equations is structurally solvable, since $V_0=V_\infty=\emptyset$. G is decomposed into 9 M-indecomposable components, V_1 through V_9, with the partial order among them depicted in Fig.8. By solving the subproblems according to this partial order, the solution to the whole system (4.1) can be obtained. Note that u_1, u_2, u_5, u_6 and u_7 are connectors and that the arc (u_2,u_1) does not belong to G_3, the subgraph corresponding to V_3. It should be remarked that V_1 through V_8 are merged into one in the L-decomposition.

In the standard form (1.1) of equations, the output variables [19], i.e., the unknowns u_k on the left-hand side, may be chosen arbitrarily to some extent. For example, the equations g_1, g_6 and g_7 in (4.1) may

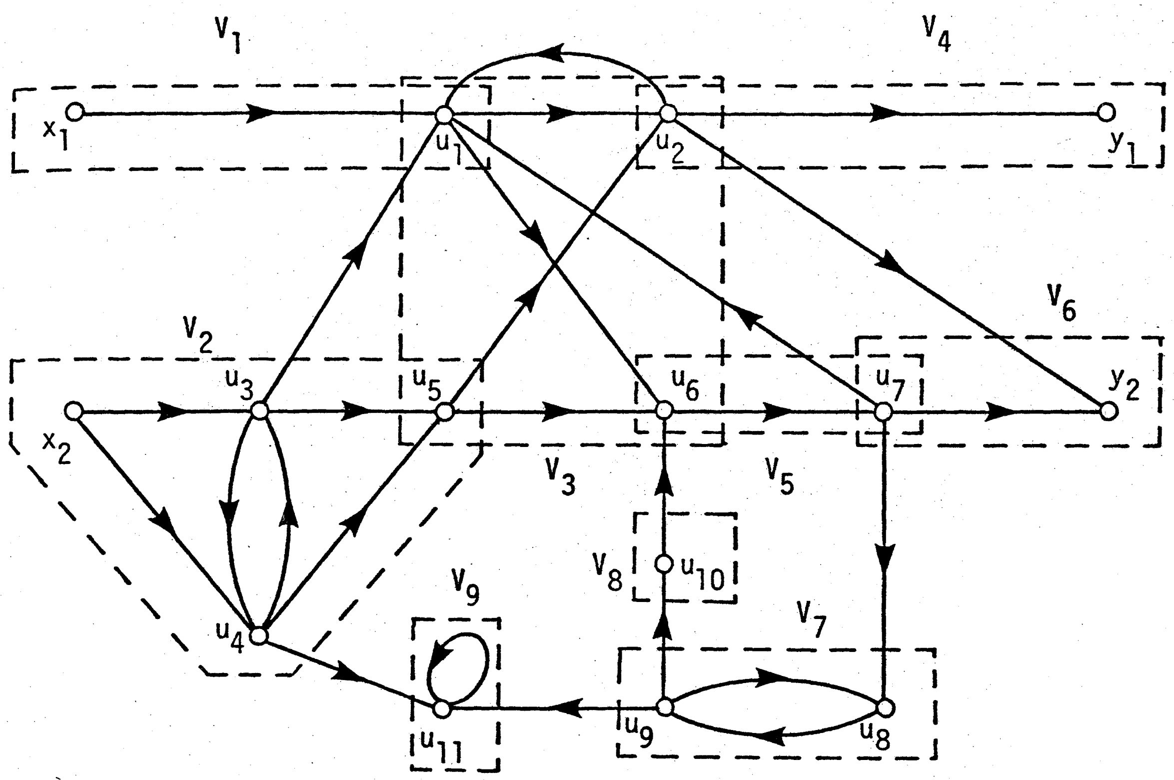

Fig.7. The representation graph G of example (4.1) and its M-decomposition ($V_0=V_\infty=\emptyset$)

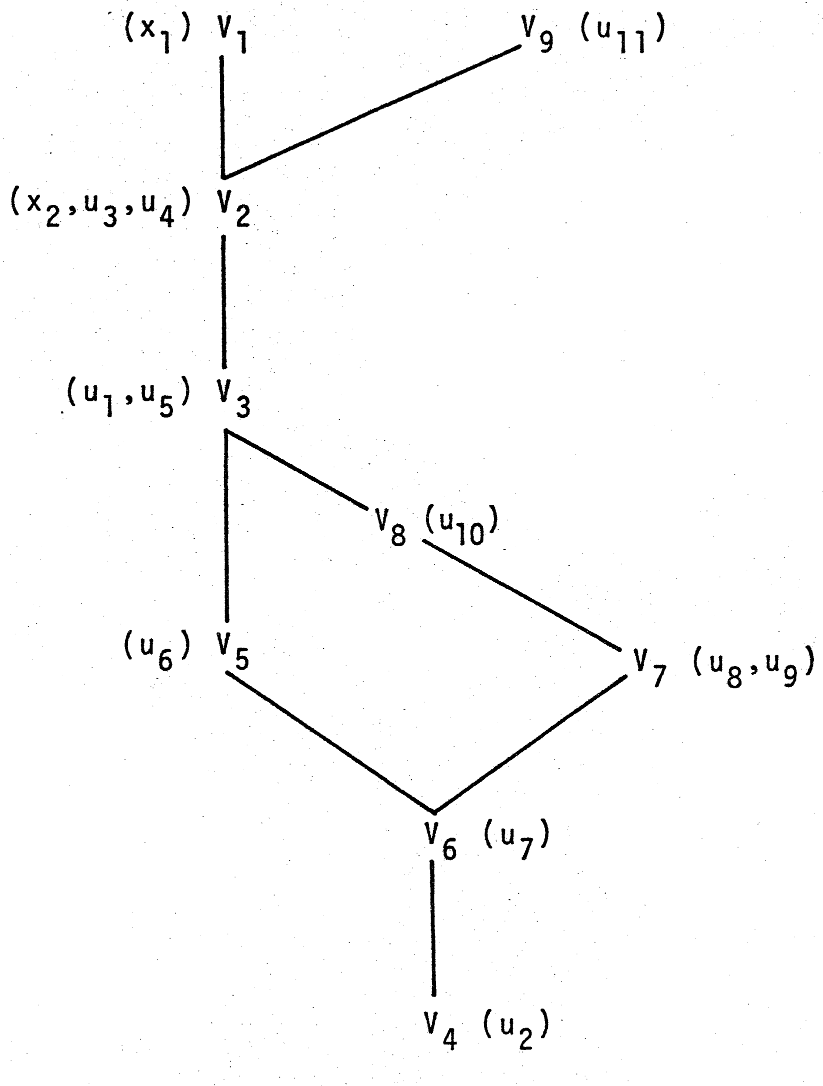

Fig.8. The Hasse diagram representing the partial order of the M-decomposition for example (4.1)

( ): Inherent unknowns of the corresponding subproblem

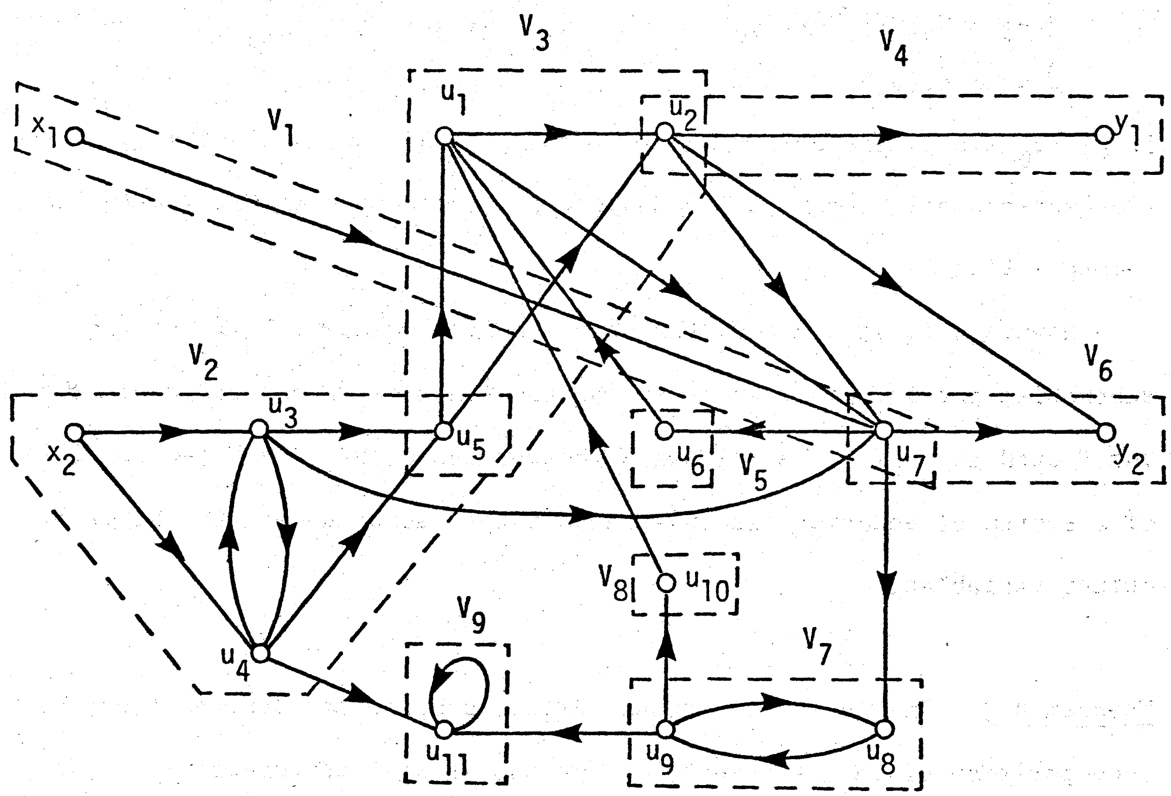

Fig.9. The representation graph G' with its M-decomposition for example (4.1) with a different set of output variables ($V_0=V_\infty= \emptyset$)

alternatively be put as

$$\begin{cases} u_1 = g_1(x_1, u_2, u_3, u_7) \\ u_6 = g_6(u_1, u_5, u_{10}) \\ u_7 = g_7(u_6) \end{cases} \Longrightarrow \begin{cases} u_7 = \tilde{g}_7(x_1, u_1, u_2, u_3) \\ u_1 = \tilde{g}_1(u_5, u_6, u_{10}) \\ u_6 = \tilde{g}_6(u_7) \end{cases}$$

if g_1, g_6 and g_7 are easily solved for u_7, u_1 and u_6, respectively. Then the representation graph G is changed to G', as shown in Fig.9 with its M-decomposition.

However, it is observed that the inherent unknowns $V_j - Y_j$ of each subproblem are invariant and that the partial order among subproblems being unaffected as well. The following theorem shows that the M-decomposition of a system of equations is invariant in this sense under the change of output variables.

<u>Theorem 4.3</u>. The inherent unknowns of the subproblems derived from the M-decomposition of the representation graph, as well as the partial order among the subproblems, are independent of the choice of output variables. ▯

<u>Proof:</u> Since the M-decomposition of a graph agrees with the DM-decomposition of the associated bipartite graph (Theorem 3.1), and since the change in the choice of output variables corresponds to the permutation of the rows of the Jacobian matrix of (1.1), followed by scaling. Q.E.D.

4.2. Cycles on the representation graph

As described in [13], [14], [20], part of the variables in (1.1) can be virtually eliminated by evaluating the functions f_i and g_k according to the structure of the representation graph. In the case where the representation graph is acyclic, the values of u_k and y_i can be computed by successive evaluation of the functions, once the values of x_j (j=1,...,N) are given; in particular, u_k (k=1,...,K) can be regarded as intermediate variables and not essential unknowns. Then the number of unknowns virtually reduces from N+K to N.

In the general case where the representation graph contains directed cycles, it has conventionally been considered that each cycle stands for an equation, which is "to be solved" by some iterative method or other. By choosing a set of variables w_d (d=1,...,D) in U (called variables of <DD> type in [14], [20]) such that every cycle on G contains at least one w_d, we obtain the <u>reduced</u> <u>system</u> of equations of the form

$$\begin{cases} y_i = F_i(x,u) & (i=1,\dots,M), \\ w_d = G_d(x,u) & (d=1,\dots,D) \end{cases} \tag{4.2}$$

with (N+D) <u>essential</u> <u>unknowns</u> x_j (j=1,...,N) and w_d (d=1,...,D). In (4.2), F_i and G_d are functions computable by successive straightforward evaluation of f_i (i=1,...,M) and g_k (k=1,...,K) in an appropriate order. Thus we may solve (4.2) by an iterative method, e.g., by the Newton method.

Consider Example (4.1), specifically the L-indecomposable component of the union of V_1 through V_8. At least three variables, e.g., u_1, u_3 and u_8, of <DD> type are necessary in order to cut all the cycles in that part of G in Fig.7. Then the reduced system has five essential unknowns x_1, x_2, u_1, u_3 and u_8. On the other hand, if we solve V_1 to V_8 separately on the basis of the M-decomposition, we have only to introduce one variable of <DD> type for each of the subproblems corresponding to V_2 and V_7; e.g., u_3 for V_2 and u_8 for V_7. Then the number of essential unknowns is equal to two in V_2 and V_3; one in V_1, V_4, V_5, V_6 and V_7; and zero in V_8. Thus each of the reduced systems for the eight subproblems contains at most two essential unknowns.

Here we will take notice of the cycle on G consisting of u_1, u_6 and u_7. In solving (4.1) on the basis of the M-decomposition, no variable of

<DD> type is necessary to cut this cycle. In other words, it may be said that this cycle does not stand for an essential equation "to be solved". Besides this cycle, we may observe that the cycle consisting of u_6, u_7, u_8, u_9 and u_{10} and that of u_1 and u_2 are of the similar kind.

As opposed to the above, the cycle composed of u_3 and u_4 contained in the subgraph G_2 for V_2 is a cycle that cannot be broken up in any decomposition that preserves the structural solvability and may be regarded as representing an essential equation. Also of this kind is the cycle of u_8 and u_9 in V_7, as well as the self-loop at u_{11} in V_9.

As illustrated above, the cycles on the representation graph can be classified into two according as they pass through a connector of the M-decomposition or not. Such cycles that contain no connectors correspond to essential equations "to be solved". We will name them essential cycles. Note that an essential cycle is contained in a subgraph G_j for a single M-component V_j.

The two kinds of cycles are not distinguished in the L-decomposition, since a strongly connected component of G is contained in an L-indecomposable component. Consequently, more variables of <DD> type (u_1, in the example above) must be introduced than is really necessary, increasing the number of essential unknowns in the reduced system of equations.

Let us consider an M-indecomposable component that is structurally solvable. In the following, we assume that (1.1) itself is M-indecomposable and structurally solvable. For each of the variables w_d ($d=1,\ldots,D$) of <DD> type, the representation graph is conceptually modified as in Fig.10 with a new maximal vertex x_{N+d} and a new minimal vertex y_{M+d}; the arcs going out of w_d in the original graph leave from x_{N+d} in the modified graph and the new arcs (w_d, y_{M+d}) and (x_{N+d}, y_{M+d}) are introduced. The entrance X and the exit Y are accordingly modified to $X \cup \{x_{N+1}, \ldots, x_{N+D}\}$ and $Y \cup \{y_{M+1}, \ldots, y_{M+D}\}$. For example, the equation $u=g(u)$ ($X=Y=\emptyset$ and $U=\{u\}$ in (1.1)) is modified to

$$\begin{cases} y = x - u, \\ u = g(x) \end{cases}$$

and value zero is set to the parameter y.

The following theorem shows that an M-indecomposable component remains M-indecomposable after the modification of this kind. In other words, a system of equations that has an M-indecomposable representation graph cannot be decomposed into subsystems even after the cycles on the representation graph are conceptually eliminated by splitting the variables of <DD> type.

Theorem 4.4. Let $G(V,E)$ ($V=X \cup U \cup Y$, $E \neq \emptyset$) be an M-indecomposable graph with entrance X and exit Y. Then the graph resulting from the modification (as in Fig.10) corresponding to variables in U of <DD> type is also M-indecomposable. ▯

Proof: Consider the associated network N_G of G (see section 2.2). When w_d is chosen as a variable of <DD> type, N_G is modified as in Fig.11 in accordance with the modification of G in Fig.10. Consider a maximal linking on G from X to Y, as well as the corresponding maximum flow f on N_G. It is not difficult to establish the theorem by inspecting the arcs on the auxiliary graph G_f for both cases where w_d is contained in the linking and where it is not. Q.E.D.

In general, the number of essential unknowns of the reduced system

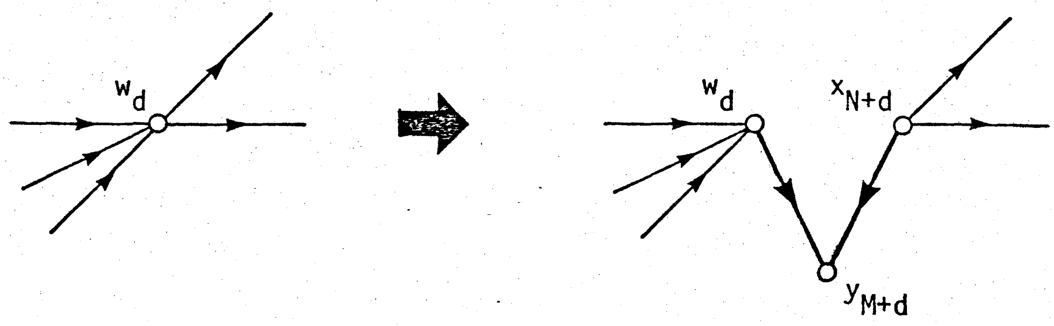

Fig.10. The modification of the representation graph for a variable of <DD> type

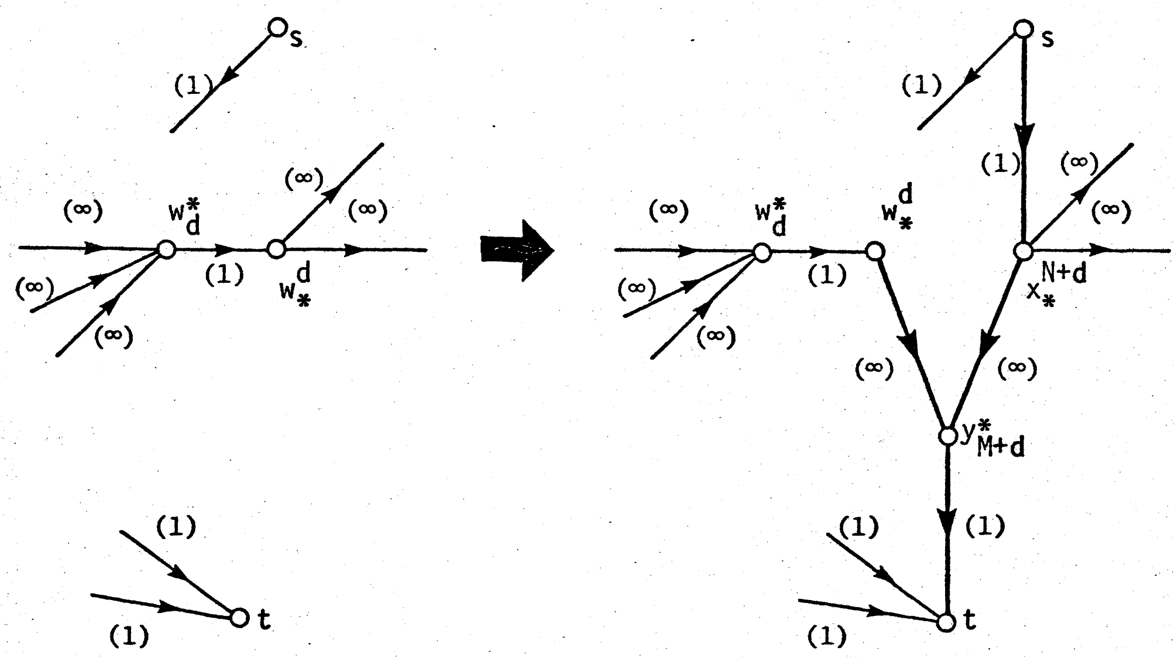

Fig.11. The modification of the associated network for a variable of <DD> type

( ): Capacity

(4.2) of equations depends on the choice of the variables of <DD> type. However, the theorem above shows that <u>the number of essential unknowns of the reduced system of an M-indecomposable system of equations is not less than the sum of the size $|X|$ of the entrance and the size of the minimum feedback vertex set of the representation graph</u>.

4.3. Decomposition of inconsistent parts

When the inconsistent parts V_0, V_∞ exist, the system (1.1) is not structurally solvable as a whole. However, the subproblems corresponding to the M-components V_j in the consistent part are structurally solvable in themselves, once the variables in $V_0 \cup V_\infty$ are fixed. In particular, such V_j as has no order relation with V_0 or V_∞ can be solved uniquely without regard to the inconsistency in V_0 and/or V_∞ (Theorem 4.2 (i)).

In this subsection, we extend the M-decomposition to investigate the structure of the inconsistent parts. To this end, we parametrize the capacity of the associated network $N_G(\tilde{V},\tilde{E},c)$ (see section 2.2) of a graph G with entrance X and exit Y:

$$c(e) = \begin{cases} 1 & e=(u^*,u_*),\ u \in U, \\ 1-a & e=(s,x_*),\ x \in X, \\ 1-b & e=(y^*,t),\ y \in Y, \\ +\infty & e=(v_*,w^*),\ (v,w) \in E \end{cases}$$

with parameters a, b (<1). (Recall that a=b=0 in section 2.2.)

Just as shown in section 2.2, the family L(a,b), for fixed a and b, of the subsets S of V which correspond to the minimum cuts constitutes a distributive lattice. Furthermore, as is evident from the theory of submodular functions [10], [11], [16],

$$L_{all} = \left(\bigcup_{0 \le a < 1} L(a,0) \right) \cup \left(\bigcup_{0 \le b < 1} L(0,b) \right)$$

is also a distributive lattice with respect to set inclusion, which determines a unique partition of $\tilde{V}$, and therefore a decomposition of V, together with the partial order, just as L=L(0,0) defines the M-decomposition. It can be shown that the decomposition of V obtained in this way is a refinement of the M-decomposition and that only the inconsistent parts V_0, V_∞ in the M-decomposition are decomposed further. Thus we may write this decomposition as

$$\{V_0^i\}_{i=1}^A \cup \{V_j\}_{j=1}^R \cup \{V_\infty^i\}_{i=1}^B \tag{4.3}$$

$$V_0 = \bigcup_{i=1}^A V_0^i, \quad V_\infty = \bigcup_{i=1}^B V_\infty^i, \tag{4.4}$$

where $\{V_0, V_\infty\} \cup \{V_j\}_{j=1}^R$ is, as usual, the M-decomposition.

For each component V_0^i (V_∞^i) in the decomposition (4.4), we define the subgraph G_0^i (G_∞^i) with entrance X_0^i (X_∞^i) and exit Y_0^i (Y_∞^i), as we did in section 2.3 for the M-decomposition. Then the following proposition holds.

<u>Proposition 4.1.</u> $|X_0^i| \geq |Y_0^i|$, $|X_\infty^i| \leq |Y_\infty^i|$. ▯

As an example, consider the graph G(V,E) in Fig.12, where $V=X \cup U \cup Y$, $X=\{x_j | j=1,...,9\}$, $U=\{u_k | k=1,...,7\}$, $Y=\{y_i | i=1,...,3\}$. The M-decomposition says that G itself is the maximal inconsistent part, while the decomposition (4.3) yields the decomposition $\{V_0^i\}_{i=1}^4$, as shown in Fig.12, with the partial order being such that $V_0^i \geqq V_0^j$ iff $1 \leq i \leq j \leq 4$. The entrances X_0^i and exits Y_0^i are given by

$$X_0^1=\{x_6,x_7,x_8,x_9\}, \quad Y_0^1=\{u_5\},$$
$$X_0^2=\{x_3,x_4,x_5\}, \quad Y_0^2=\{u_3\},$$
$$X_0^3=\{u_3,u_5\}, \quad Y_0^3=\{u_4,u_6\},$$
$$X_0^4=\{x_1,x_2,u_4,u_6\}, \quad Y_0^4=\{y_1,y_2,y_3\}.$$

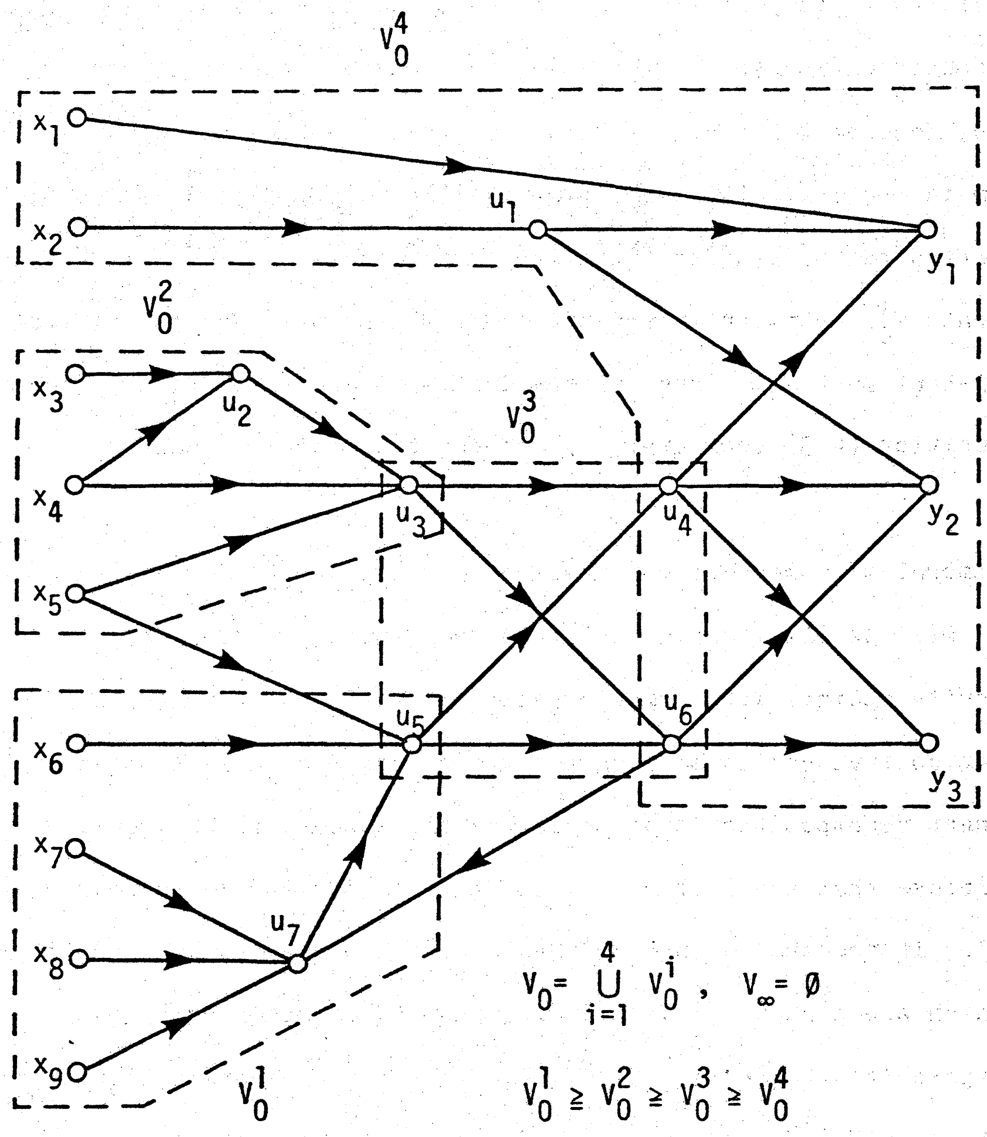

Fig.12. An example of the decomposition (4.3) for inconsistent parts

It is observed that Proposition 4.1 holds in this case.

In the case where the system (1.1) of equations is not structurally solvable, the decomposition (4.2) reveals, to some extent, the structure of the inconsistent parts. To be specific, consider the case where $V_0 \neq \emptyset$. Then, by Theorem 2.2 (iv), we have the excessive degrees $|X_0|-|Y_0|$ (>0) of freedom in the maximal inconsistent part V_0. Proposition 4.1 may be interpreted as implying that the excessive degrees are distributed over the components V_0^i. Similar interpretation could be made for the minimal inconsistent part V_∞. Thus, we may deal with each component in the decomposition (4.3) separately to correct the inconsistencies.

4.4. Amount of numerical computation

It will be illustrated here that the M-decomposition is not necessarily optimal with regard to the total amount of numerical computation involved in solving the whole system of equations, though it is the finest decomposition that preserves the structural solvability.

Suppose that the system (1.1) of equations, assumed to be structurally solvable, is decomposed into S subproblems P_j (j=1,...,S) in the standard form which are structurally solvable, though not necessarily M-indecomposable. Let $G_j(X_j \cup U_j \cup Y_j, E_j)$ be the representation graph of P_j and set $|X_j|=|Y_j|=N_j$ and $|U_j|=K_j$. The number of essential unknowns in the reduced system (4.2) for P_j is equal to N_j+D_j, where D_j denotes the (minimum) number of variables of <DD> type, or the size of the minimum feedback arc set.

We will roughly estimate the amount of numerical computation needed in solving the subproblem P_j by an iterative method, say, the Newton method. Under the assumption that each function evaluation of f_i or g_k costs about

c_1, the amount of computation of (y,w) from (x,w) in (4.2) is equal to $c_1(K_j+N_j)$. On the other hand, the inversion of the Jacobian matrix, assumed to be dense, would require $c_2(N_j+D_j)^3$ computation. Then, each Newton iteration requires

$$c_1(K_j+N_j)+c_2(N_j+D_j)^3$$

computation. If the number of iterations can be regarded as constant for all subproblems, the total amount of computation in solving the whole system (1.1) would be proportional to

$$c_1\sum_{j=1}^{S}(K_j+N_j) + c_2\sum_{j=1}^{S}(N_j+D_j)^3$$

$$= c_1(K+N) + c_2\sum_{j=1}^{S}(N_j+D_j)^3 \qquad (4.5)$$

The finer the decomposition, the larger the number of subproblems. Here we will compare the M- and the L-decomposition. First consider a system of equations with the representation graph being a cascade of complete bipartite graphs of order N, as shown in Fig.13 for N=3. This graph is L-indecomposable but is decomposed into m complete bipartite graphs of order N by the M-decomposition. The amount of computation is estimated by (4.5) as

M-decomposition: $c_1mN + c_2mN^3$,

L-decomposition: $c_1mN + c_2N^3$.

Evidently, the M-decomposition is too fine to be successful for this example.

Next example, shown in Fig.14, is again L-indecomposable, whereas it is decomposed into mN/2 bipartite graphs of order two by the M-decomposition. The estimate (4.5) yields

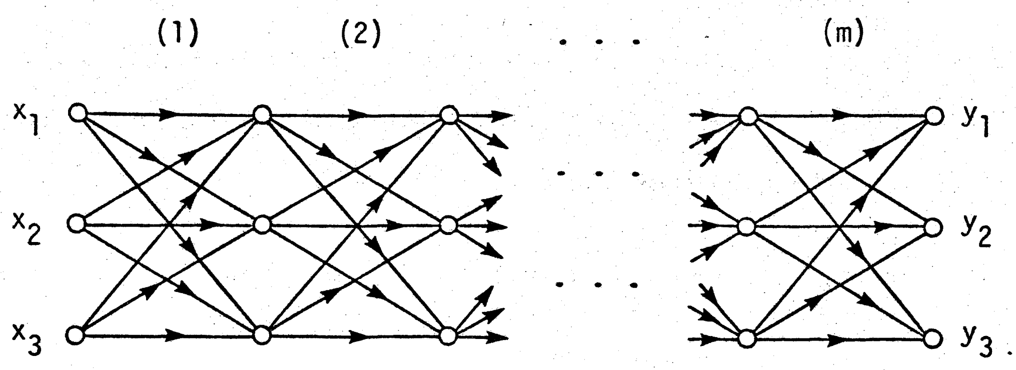

Fig.13. An example for which L-decomposition is more successful than M-decomposition

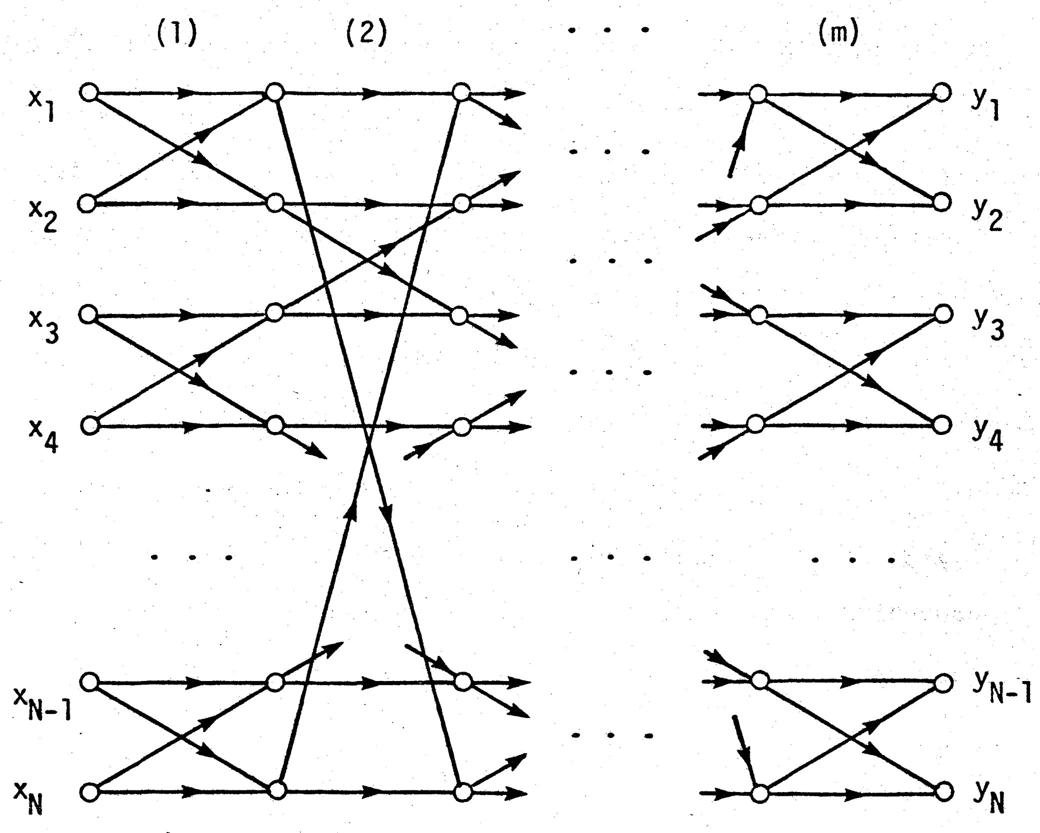

Fig.14. An example for which M-decomposition is more successful than L-decomposition (N: even)

M-decomposition: $c_1 mN + 4c_2 mN$,

L-decomposition: $c_1 mN + c_2 N^3$.

Here it may be remarked that the Jacobian matrix of the reduced system of the whole system, being L-indecomposable, is dense for m which is at least as large as N. Then the M-decomposition is definitely more advantageous.

As is seen from the examples above, neither the M-decomposition nor the L-decomposition is universally optimal in regard to the amount of numerical computation. In general, a decomposition into structurally solvable subproblems P_j are obtained as an order-homomorphic image of the M-decomposition $\{V_j\}$. For instance, if two M-indecomposable components V_1 and V_2 ($V_1 \gneq V_2$) have no components between them (i.e., such V_j as $V_1 \gneq V_j \gneq V_2$), the corresponding subproblems P_1 and P_2 may be merged into one problem P'. If we denote by N', K', D' the quantities of P' corresponding to N_j, K_j, D_j of P_j, we have the following relations:

$$N'+K' = (N_1+K_1) + (N_2+K_2),$$

$$N' \leq N_1 + N_2,$$

$$D' \geq D_1 + D_2.$$

The last inequality accounts for the possibility that additional variables of <DD> type may be necessary to cut the inessential cycles introduced as a result of merging the M-indecomposable components.

The consideration above would suggest that, in actually solving a large-scale system of equations, we should try to minimize the total amount of numerical computation by selecting an appropriate decomposition, which may be obtained by merging the M-components in the light of some relevant criterion such as (4.5).

5. Conclusion

In this paper, the M-decomposition is defined for a graph and applied to the structural analysis of a large-scale system of equations. In particular, the essential cycles on the representation graph are distinguished.

How to treat the inconsistent parts is an issue of great importance in practice. The decomposition given in section 4.3 is nothing but a possible way among many others. The theory of principal structure of a submodular system [8] would be also applicable. A concrete algorithm for finding an optimal decomposition with regard to the total amount of computation is left for future investigation.

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References

[1] Aho, A. V., Hopcroft, J. E., and Ullman, J. D.: The Design and Analysis of Computer Algorithms. Addison-Wesley, Reading, 1974.

[2] Birkhoff, G.: Lattice Theory (3rd edition). American Mathematical Society Colloquium Publications Vol.25, Providence, 1967.

[3] Dulmage, A. L., and Mendelsohn, N. S.: Coverings of Bipartite Graphs. Canadian Journal of Mathematics, Vol.10 (1958), pp.517 - 534.

[4] Dulmage, A. L., and Mendelsohn, N. S.: A Structure Theory of Bipartite Graphs of Finite Exterior Dimension. Transactions of the Royal Society of Canada, Section III, Vol.53 (1959), pp.1 - 13.

[5] Dulmage, A. L., and Mendelsohn, N. S.: Two Algorithms for Bipartite Graphs. SIAM Journal, Vol.11 (1963), pp.183 - 194.

[6] Even, S., and Tarjan, R. E.: Network Flow and Testing Graph Connectivity. SIAM Journal on Computing, Vol.4 (1975), pp.507 - 518.

[7] Ford, L. R., Jr., and Fulkerson, D. R.: Flows in Networks. Princeton University Press, Princeton, 1962.

[8] Fujishige, S.: Principal Structure of Submodular Systems. Discrete Applied Mathematics, Vol.2 (1980), pp.77 - 79.

[9] Iri, M.: Network Flow, Transportation and Scheduling. Academic Press, New York, 1969.

[10] Iri, M.: A Review of Recent Work in Japan on Principal Partitions of Matroids and Their Applications. Annals of the New York Academy of Sciences, Vol.319 (1979), pp.306 - 319.

[11] Iri, M., and Fujishige, S.: Use of Matroid Theory in Operations Research, Circuits and Systems Theory. International Journal of Systems Science, Vol.12 (1981), pp.27 - 54.

[12] Iri, M., and Han, T.-S.: Linear Algebra (in Japanese). Kyoiku Shuppan, Tokyo, 1977.

[13] Iri, M., Tsunekawa, J., and Murota, K.: Graph-Theoretic Approach to Large-Scale Systems --- Structural Solvability and Block-Triangularization (in Japanese). Transactions of Information Processing Society of Japan, Vol.23 (1982), pp.88 - 95. (English translation available).

[14] Iri, M., Tsunekawa, J., and Yajima, K.: The Graphical Techniques Used for a Chemical Process Simulator "JUSE GIFS". Information Processing 71 (Proceedings of the IFIP Congress 71), Vol.2 (Applications), 1972, pp.1150 - 1155.

[15] Nakamura, M., and Iri, M.: Poly-Linking Systems and Principal

Partitions (in Japanese). Proceedings of Spring Conference of the Operations Research Society of Japan, B-6, pp.56 - 57, 1980.

[16] Nakamura, M., and Iri, M.: A Structural Theory for Submodular Functions, Polymatroids and Polymatroid Intersections. (1981). Submitted for publication.

[17] Picard, J. C., and Queyranne, M.: On the Structure of All Minimum Cuts in a Network and Applications. Mathematical Programming Study, Vol.13 (1980), pp.8 - 16.

[18] Schrijver, A.: Matroids and Linking Systems. Mathematical Centre Tracts, Vol.88, Amsterdam, 1978.

[19] Steward, D. V.: On an Approach to Techniques for the Analysis of the Structure of Large Systems of Equations. SIAM Review, Vol.4 (1962), pp.321 - 342.

[20] Tsunekawa, J.: A Graph-Theoretic Partition of a Large-Scale System (in Japanese). Communications of the Operations Research Society of Japan, Vol.25 (1980), pp.788 - 793.