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A SURVEY OF RESULTS ON THE INVERSES OF TOEPLITZ AND BAND MATRICES

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ABSTRACT

This survey mentions some of the efforts which have been made recently in finding the inverses of Toeplitz and band matrices. Many authors have concentrated on properties of the inverses in cases where finding the actual elements of the inverse seems too difficult. Some variations on the strict Toeplitz form, all elements identical on every diagonal, and on the fixed bandwidth form are included.

1. INTRODUCTION

Toeplitz and band matrices arise in many problems. The inverses of these matrices and the properties of their inverses are useful. The best survey of the recent work is by Roebuck and Barnett [29]. Trench [34] reports that Toeplitz matrices play an important role in discrete random processes. Fischer and Usmani [7] show how to find accurate error bounds on approximations to solutions of some boundary value problems. They use the infinity norm of the inverse of a tridiagonal matrix.

Toeplitz matrices have the same element repeated on each diagonal. An example of a Toeplitz matrix of bandwidth k+l+1 is

$$\begin{pmatrix} a_o & a_1 & \cdots & a_k & & 0 \\ a_{-1} & a_o & & & \ddots & \\ \vdots & & \ddots & & & a_k \\ a_{-\ell} & & & \ddots & & \vdots \\ & \ddots & & & \ddots & \\ 0 & & a_{-\ell} & \cdots & & a_o \end{pmatrix}_{n\times n} . \quad (1\cdot 1)$$

Variations which have irregular corner elements are included in this survey.

A circulant matrix is a Toeplitz matrix but would not usually be considered a band matrix. An example of a

circulant is

$$\begin{pmatrix} a_o & a_1 & \cdots & a_k & & a_{-\ell} & \cdots & a_{-1} \\ a_{-1} & a_o & & \ddots & & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & & \ddots & a_{-\ell} \\ a_{-\ell} & & \ddots & \ddots & & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & & \ddots & a_k \\ a_k & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ a_1 & \cdots & a_k & a_{-\ell} & \cdots & & & a_o \end{pmatrix}_{n\times n} . \quad (1\cdot 2)$$

The various papers included here are classified according to the type of matrix which is inverted or studied. The first group is the tridiagonal type. These matrices can easily be inverted even when not of Toeplitz type. The second group is the five-band type. All the results in this group are for Toeplitz or nearly Toeplitz matrices. Most authors further restrict themselves to symmetric matrices. The third group is the Toeplitz fixed bandwidth type where the elements of the matrix are special numbers or have special properties. Finally, the fourth group is the general Toeplitz band type. There is a variety of techniques used here and the form of the answer varies a great deal from paper to paper.

2. TRIDIAGONAL FORM

Many authors have found the inverses of tridiagonal or slight variations of tridiagonal matrices with and without the restriction of Toeplitz form. The general appearance of such matrices is

$$\begin{pmatrix} b_1 & c_1 & & & & d_1 \\ a_2 & b_2 & c_2 & & 0 & \\ & a_3 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & c_{n-1} \\ d_n & & & & a_n & b_n \end{pmatrix}_{n\times n} . \qquad (2\cdot 1)$$

Tridiagonal form would have zero in the lower left and upper right corners. It is difficult to make a complete list but a representative set of papers is collected here in an approximate order from the most general matrix first to the most specific matrix last. It may seem strange that so many papers have been written on the same problem. The reason is that slight specializations in the problem make tremendous changes in the formulae for the inverse and in how explicit a formula can be given for the inverse. The references give the author's name, year of publication and a description of the matrix treated. Some comments on the method used and the form of the answer are included.

Rack (1978/9) inverts the general matrix (2.1), [27]. A three term recurrence must be solved before the formula for the elements of the inverse can be written explicitly. The tridiagonal is treated as a special case and the elements of its inverse are written in terms of Lucas polynomials.

Torii (1966) and Ikebe (1979) both invert the tridiagonal form, [32],[33],[16],

$$d_1 = d_n = 0 .$$

Torii expresses elements of the inverse in terms of fundamental solutions of two difference equations. Ikebe inverts a general Hessenberg matrix and applies the result to a tridiagonal matrix. He shows that each element of the inverse is an inner product of two vectors. In the proof, the matrix is partitioned a certain way to give information about the inverse.

Kershaw (1970) finds upper and lower bounds on the elements of the inverse of a tridiagonal in which the super-diagonal is related to the sub-diagonal, [18],

$$d_1 = d_n = 0; \; c_i = 1 - a_i, \; i = 2, 3, \ldots , n - 1.$$

The bounds are easily calculated from the elements of the tridiagonal.

Clement (1959) inverts a tridiagonal which has zeros on its diagonal, [4],

$$d_1 = d_n = 0; \; b_i = 0, \; i = 1, 2, \ldots , n .$$

The inverse exists for even n and many of its elements are

zero. The non-zero elements are given as a quotient of products of the elements of the tridiagonal.

Schlegel (1970) inverts a symmetric, persymmetric tridiagonal, [30],

$$d_1 = d_n = 0;$$
$$a_i = 1,\ i = 2, 3, \ldots, n;$$
$$c_i = 1,\ i = 1, 2, \ldots, n-1;$$
$$b_i = b_{n-i+1},\ i = 1, 2, \ldots, [n/2] .$$

After n values are calculated from a three term recurrence, each element of the inverse can be written as a product of three of these values.

Searle (1979) inverts a three term circulant, [31],

$$a_i = a,\ i = 2, 3, \ldots, n; \qquad d_1 = a;$$
$$b_i = b,\ i = 1, 2, \ldots, n;$$
$$c_i = c,\ i = 1, 2, \ldots, n-1; \quad d_n = c .$$

Explicit formulae for the elements of the inverse are given. Since any circulant can be written as a product of three term circulants, these results provide an algorithm for inverting any circulant. The formulae are obtained by solving a three term constant coefficient difference equation.

Uppuluri and Carpenter (1970) invert a Toeplitz tridiagonal matrix, [36],

$$d_1 = d_n = 0;$$
$$a_i = a,\ i = 2, 3, \ldots, n;$$
$$b_i = b,\ i = 1, 2, \ldots, n;$$
$$c_i = c,\ i = 1, 2, \ldots, n-1.$$

They find elements of the inverse by solving a three term constant coefficient difference equation. The elements of the inverse are expressed in terms of elements of the powers of a certain 2 x 2 matrix.

Valvi (1977) inverts a symmetric tridiagonal which depends on two parameters, [38],

$$d_1 = d_n = 0;$$
$$a_i = a,\ i = 2, 3, \ldots, n;$$
$$c_i = c,\ i = 1, 2, \ldots, n-1;$$
$$b_1 = a,\ b_n = a;\ b_i = a + c,\ i = 2, 3, \ldots, n-1.$$

The elements of the inverses are given explicitly. This paper also gives the inverses of two other special matrices.

Meek (1980) inverts a matrix which is symmetric, Toeplitz

and tridiagonal except for the corner elements, [23],

$$a_i = -1,\ i = 2, 3, \ldots, n;$$
$$b_i = b,\ i = 1, 2, \ldots, n;$$
$$c_i = -1,\ i = 1, 2, \ldots, n-1.$$

The matrix is partitioned to separate the corners from the more regular central portion. The solution of a three term constant coefficient difference equation is needed to express the elements of the inverse. Some conditions that the inverse is positive are given.

Kershaw (1969) inverts two symmetric matrices. One is Toeplitz and tridiagonal, except for corner elements, and the other is a three term circulant, except for corner elements, [17],

$$d_1 = d_n = 0;$$
$$a_i = 1,\ i = 2, 3, \ldots, n;$$
$$b_i = b,\ i = 2, 3, \ldots, n-1;$$
$$c_i = 1,\ i = 1, 2, \ldots, n-1;$$

and

$$d_1 = d_n = 1;$$
$$a_i = 1,\ i = 2, 3, \ldots, n;$$
$$b_1 = b_n;\ b_i = b,\ i = 2, 3, \ldots, n-1;$$
$$c_i = 1,\ i = 1, 2, \ldots, n-1.$$

The formulae are obtained from difference equation considerations and are expressed in terms of Chebyshev polynomials.

Fischer and Usmani (1969) invert a symmetric Toeplitz tridiagonal matrix, [7],

$$d_1 = d_n = 0;$$
$$a_i = -1,\ i = 2, 3, \ldots, n;$$
$$b_i = b,\ i = 1, 2, \ldots, n;$$
$$c_i = -1,\ i = 1, 2, \ldots, n-1.$$

The method used is Gaussian elimination and the elements of the inverse are expressed in terms of the solution of a three term constant coefficient difference equation. The norm of the inverse is calculated and used in an error analysis of the approximate solution of a boundary value problem.

Ford (1975) inverts a specific three term circulant which arises from periodic cubic spline interpolation, [8],

$$d_1 = d_n = 1;$$
$$a_i = 1,\ i = 2, 3, \ldots, n;$$
$$b_i = 4,\ i = 1, 2, \ldots, n;$$
$$c_i = 1,\ i = 1, 2, \ldots, n-1.$$

He finds limiting values, as n becomes large, for the elements of the inverse matrix. It is then easy to solve a set of linear equations involving this matrix.

3. TOEPLITZ FIVE-BAND FORM

Five-band matrices are significantly harder to invert than tridiagonal matrices. A few authors have studied symmetric Toeplitz five-band matrices with some anomolous corner elements. Even this restricted class of matrices is fairly difficult and most papers give properties of the inverse matrix, not explicit formulae for the elements. The general matrix of this type is

$$\begin{pmatrix} A & b & c & & & & B \\ b & a & b & c & & 0 & \\ c & b & a & \ddots & \ddots & & \\ & c & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & c & \\ & & & & a & b & \\ C & & & & c & b & D \end{pmatrix}_{n\times n} . \qquad (3\cdot1)$$

The references listed below are in an approximate order from the most general matrix first to the most specific matrix last.

Meek (1980) inverts (3.1), [22]. The method is to partition the matrix in order to separate the corner elements from the more regular central portion. The formula for elements of the inverse depends on the solution of a fourth-order constant coefficient difference equation.

Meek (1976) inverts a Toeplitz five-band matrix, [21],

$$A = 1,\ B = 0,\ C = 0,\ D = 1,\ a = 1 .$$

The top row of the inverse is found and a condition that the entire inverse is positive is given. A fourth-order constant coefficient difference equation is used to express elements of the top row of the inverse and partitioning is used to relate them to elements elsewhere in the matrix.

Hoskins and McMaster (1976,1977,1978) discuss properties of the inverse of a symmetric Toeplitz five-band matrix, [12], [13], [15],

$$A = a,\ B = 0,\ C = 0,\ D = a,\ c = 1 .$$

The matrix is expressed as a product of matrices of smaller bandwidth. Properties such as the sign pattern of the inverse and bounds on the infinity norm of the inverse are studied. Some variations on the Toeplitz form are included.

Usmani (1971) inverts a symmetric five-band matrix with one parameter, [37],

$$A = a - 1,\ B = 0,\ C = 0,\ D = a - 1$$
$$c = -1,\ b = -\frac{a}{2} + 1 .$$

He solves a fourth-order constant coefficient difference equation to find the elements of the inverse. A simple condition that the inverse is positive is given.

4. SPECIAL TOEPLITZ FORMS OF UNRESTRICTED BANDWIDTH

A few Toeplitz matrices of unrestricted bandwidth have been inverted explicitly. The methods are generally combinatorial in nature and rely heavily on the specialized form and elements of the matrix which is being inverted. The papers are listed in alphabetical order by author's last name.

Andres, Hoskins and Stanton [2] invert a class of skew-symmetric Toeplitz band matrices whose coefficients are derived from finite difference operators. An example of the matrices treated is

$$\begin{pmatrix} 0 & 2 & -1 & & & & \\ -2 & 0 & 2 & -1 & & 0 & \\ 1 & -2 & 0 & \ddots & \ddots & & \\ & 1 & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & -1 \\ & 0 & & \ddots & \ddots & \ddots & 2 \\ & & & & 1 & -2 & 0 \end{pmatrix}_{n\times n} .$$

Hoskins and Meek [14] find an attainable bound on the infinity norm of a class of circulant matrices. This class includes matrices which arise from periodic polynomial spline interpolation. Kershaw [19] gives the bound for those matrices which arise from odd order periodic spline interpolation.

Hoskins and Ponzo [11] invert a class of symmetric Toeplitz band matrices whose entries are binomial coefficients. An example of the class of matrices treated is

$$\begin{pmatrix} 6 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & 0 & \\ 1 & -4 & 6 & \ddots & \ddots & & \\ & 1 & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & 1 \\ & 0 & & \ddots & \ddots & \ddots & -4 \\ & & & & 1 & -4 & 6 \end{pmatrix}_{n\times n} .$$

Rehnqvist [28] inverts the symmetric Toeplitz band matrix

$$\begin{pmatrix} k & k-1 & k-2 & \dots & 1 & & \\ k-1 & k & \ddots & \ddots & \ddots & 0 & \\ k-2 & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & & & \ddots & \ddots & \ddots & \vdots \\ & 0 & & & \ddots & \ddots & k-1 \\ & & & & 1 & \dots & k-1 \quad k \end{pmatrix}_{n\times n} .$$

5. GENERAL TOEPLITZ BAND FORM

Many authors have found formulations for the inverses of Toeplitz band matrices without restricting the bandwidth or assuming any special coefficients in the matrix. The many papers which treat the solution of linear equations which have a Toeplitz or band matrix will be omitted here in favour of those papers which attempt more explicit formulations for the inverse matrix. While most of the results are for Toeplitz band matrices, three papers treat more general cases. Yamamoto and Ikebe [39] and Oohashi [26] invert a band matrix without assuming Toeplitz form and Trench [34] inverts a Toeplitz matrix without assuming a fixed bandwidth. Another paper [9] expresses the inverse of Toeplitz and nearly Toeplitz matrices by means of recursions. The papers listed below are in alphabetical order by author's last name.

Allgower [1] inverts a symmetric Toeplitz band matrix of bandwidth $2k-1$. He proves a useful result that it suffices to find only $k(k-1)/2$ elements of the inverse. The remaining elements can be computed by elementary calculations. Unfortunately, the process of finding those $k(k-1)/2$ elements can be rather difficult. The cases $k=3$ and $k=4$, five-band and seven-band matrices, are discussed in detail. A constant coefficient difference equation of order $2k-1$ must be solved after which the elements of the inverse can be calculated with a simple formula. Many examples of the inverses of special five and seven-band matrices are given.

Bareiss [3], while not actually looking for the inverse, gives an interesting formulation for the solution of a linear system whose matrix is Toeplitz. His method involves recurrences with determinants. The formulae in Meek [24] are similar to those found by Bareiss.

Meek [24] inverts a Toeplitz band matrix of bandwidth k and with s super-diagonals. A constant coefficient difference equation of order k must be solved before the elements of the inverse can be written. Each element of the inverse is an s+1 x s+1 determinant divided by an s x s determinant. Thus, the formulae are simple if there are a small number of super-diagonals. Some conditions on the solution of the difference equation that imply the inverse matrix is positive are given for small s.

Mentz [25] inverts a symmetric Toeplitz band matrix of bandwidth 2k-1. The matrix is first written as a linear combination of k simple matrices whose elements are all zeros except for at most two symmetrically placed diagonals of ones. The inverse can be expressed as the solution of a system of linear difference equations. The cases k=2 and k=3, tridiagonal and five-band matrices, are treated in detail and there is some discussion on the elements of the inverse for large n.

Oohashi [26] inverts a band matrix of bandwidth k+l+1 and expresses the elements of the inverse in terms of the solutions of some linear difference equations of order k+l.

Each element of the inverse is an inner product of two vectors.

Trench [34] inverts a positive definite Toeplitz matrix. A partitioning of the original matrix by separating the left column and the top row shows how the inverse of an n x n matrix is related to the inverse of an n^{-1} x n^{-1} matrix. This recursion is then simplified considerably to give an algorithm to find the inverse of just the n x n matrix.

Trench [35] inverts a Toeplitz band matrix of bandwidth k. The same partitioning is used as in [34]. A fixed bandwidth means that some of the recursions in the earlier work can be solved in terms of the solutions of linear difference equations. Two constant coefficient difference equations of order k-1 must be solved. Some examples are worked and an algorithm is given.

Yamamoto and Ikebe [39] invert a band matrix of bandwidth k. The formula for the elements of the inverse is essentially a sum of products of determinants. The determinants are of order k^{-1} x k^{-1} and s x s where s is the number of super-diagonals and their entries are produced from k^{-1} recurrences. The development of the formulae is based on difference equations. The general tridiagonal is given as an example.

6. PROPERTIES OF INVERSE IN GENERAL CASE

This short section will survey papers which find properties of inverses of Toeplitz or band matrices without finding the elements of the inverse.

Demko [5], [6] gives some results for the inverses of band matrices. The first paper shows that the elements in each diagonal of the inverse of a diagonally dominant matrix are bounded by quantities which exponentially decay as one moves away from the main diagonal. The second paper gives a method to bound the infinity norm of the inverse of a band matrix.

Grenander and Szego [10] are not concerned with inverses but give so many interesting theorems on the location of eigenvalues of Toeplitz matrices that it would be a shame not to mention their work here.

Lorenz and Mackens [20] give a condition that the inverse of a Toeplitz matrix of bandwidth k is totally non-negative, which means that all of its minors are non-negative. The condition is that the zeros of a polynomial of degree k are positive or 'nearly' positive.

7. CONCLUSIONS

Results on the inverses of Toeplitz and band matrices have aroused considerable interest. However, it appears that the results of one worker are not known to another so that many results are rediscovered. This survey has attempted to

collect the various papers and give brief summaries of their results. It is hard to say at present which results will be most useful as many are very complicated algebraically.

One direction which needs more attention is the inversion of matrices which are slight variations on Toeplitz band matrices. These matrices arise naturally in boundary value problems and appear to have properties, like the infinity norm of the inverse, which are very close in value to the corresponding property of the Toeplitz band matrix.

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