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Recent Developments in the Theory of Large N Gauge Fields

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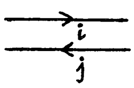
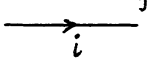
It has been known for some time that quantum field theories with global or local U(N) (SU(N), O(N),,,) symmetry greatly simplify in the limit $N\to\infty$. The well-known example is the so-called O(N) vector model which is a theory of massless scalar fields ϕ_i (i=1,,N) taking values on S^{N-1}. In two dimensions lagrangian is given by

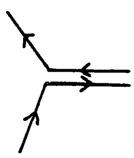
$$\mathcal{L} = \int \frac{1}{2} \sum_{i=1}^{N} (\partial_\mu \phi_i)^2 \, d^2x , \qquad \sum_{i=1}^{N} \phi_i^{\,2} = 1. \tag{1}$$

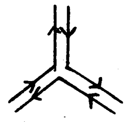
It is known that the interaction caused by the constraint $\sum\phi_i^{\,2}=1$ creates a finite mass gap in this system ($N \geq 3$). In fact the model becomes exactly soluble in the large N limit and becomes a free theory of massive scalar particles. When N is finite, these massive fields begin to interact weakly with the strength 1/N. Thus the $N = \infty$ limit yields an exact solution which gives a qualitatively correct description of the system also for finite N.

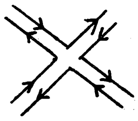
In the case of gauge theories it is also known that a considerable simplification takes place in the limit of large gauge group. Prior to the recent developments which I am going to describe the following characteristic properties have been known of the $N = \infty$ gauge fields.

1. The planar dominance of weak coupling perturbation theory;[1]
2. The factorization property of Wilson loop amplitudes.[2]

Properties 1 and 2 are derived from a simple power-counting analysis of Feynman diagrams. Suppose we have an U(N) gauge field interacting with a quark belonging to the fundamental representation of the gauge group. Being in an adjoint representation gauge fields carry a pair of color indices (i,j) while a quark field carries a single index i. Then a gluon propagator is represented by a pair of lines going into opposite directions  while a quark propergator is represented by a line  . Interaction vertices of the theory are given by

 ; quark-gluon vertes, strength g

 ; 3-gluon vertex, g

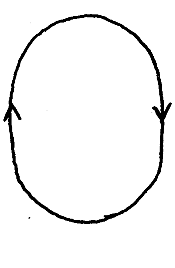
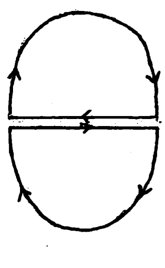
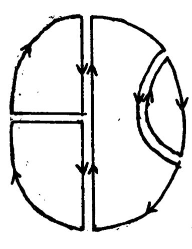
 ; 4-gluon vertex, g^2

Here g is the coupling constant. A Feynman diagram aquires a weight $g^{V+V_3+2V_4} \times N^{\ell}$ when there exist V quark-gluon vertices, V_3 3-gluon vertices, V_4 4-gluon vertices and ℓ closed color loops in the diagram. Then a simple combinatorial analysis shows that when we let g^2N to be independent of N (or let $g^2 = O(1/N)$), the weight of a diagram becomes N^{χ} where χ is the Euler characteristic of the diagram (interaction vertices, propagators and color loops are regarded as vertices, edges and faces of a polyhedron, respectively). Thus in the limit of $N = \infty$ with g^2N fixed graphs with the heighest Euler number give the dominant contributions and we may ignor all other types of diagrams.

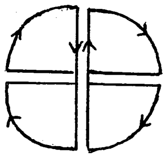
In the case of Feynman diagrams with a single quark loop C which contribute to the Wilson loop amplitude

$$\langle \exp i \oint_C A_\mu dx_\mu \rangle \tag{2}$$

the leading contributions come from the planar graphs with the topology of a disc,

 ,  ,  , etc.

None-planar graphs like



are down by N^{-2h} where h is the number of handles in the diagram. Thus we need to consider only planar diagrams in the computation of Wilson loop amplitude.

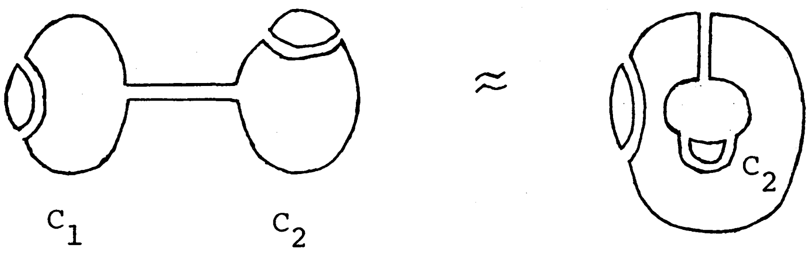
Moreover when we consider the case of more than one quark loop, say C_1 and C_2, the leading contribution to the correlation function

$$<\exp i \oint_{C_1} A_\mu dx_\mu \exp i \oint_{C_2} A_\mu dx_\mu> \qquad (3)$$

come from the disconnected piece,

$$<\exp i \oint_{C_1} A_\mu dx_\mu> <\exp i \oint_{C_2} A_\mu dx_\mu> \approx O(N^2). \qquad (4)$$

This is because when there exist gluon exchanges between C_1 and C_2 we obtain a topology of an annulus,



and hence of order only $O(N^0)$.

In this way in the large N limit with g^2N fixed there exist no correlation between quark loops and the amplitudes always factor

$$\langle \prod_i \exp i \oint_{C_i} A_\mu dx_\mu \rangle = \prod_i \langle \exp i \oint_{C_i} A_\mu dx_\mu \rangle \qquad (5)$$

Since a Wilson loop amplitude may be interpreted as a meson propagator, Eq.(5) implies that at $N = \infty$ gauge interactions are exhausted to form bound states and mesons do not scatter from each other. In this respect $N = \infty$ gauge theory is an ideal place to look at the confining property of gauge fields and here also lies our hope for their analytic solutions.

Let us now turn to the discussion of recent developments on large N gauge and spin systems.[3,4,5,6] As a result of these investigations we now have a further characterization of $N = \infty$ gauge fields;

3. Reduction of dynamical degrees of freedom.

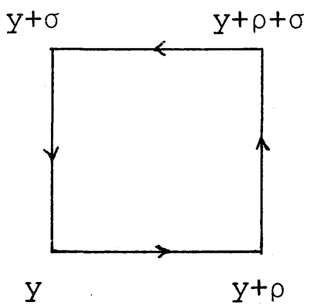
Property 3 means that one may replace the $N = \infty$ gauge field theory by a much simpler system, a model with only a finite number (=space-time dimensionality) of U(N) matrices, without loosing information of the original theory. This is a remarkable result in the sense that a quantum field theory may be reduced to a kind of dynamical system with a finite number of dynamical variables.

In the following we present the original argument for reduction[3] using the lattice formulation of gauge fields. In the lattice gauge theory[7] the basic dynamical variables are link variables defined on each link of the d-dimensional space-time lattice. A link variable $U_{x,\mu}$ is an U(N) matrix lying on the link connecting x and x + μ (μ is the unit vector in the μ-direction). The correspondence to the continuum theory is given by

$$U_{x,\mu} \approx \exp iga A_\mu(x) \qquad (6)$$

where a is the lattice constant. The action of the standard Wilson theory is defined by

$$S = - \sum_{y} \sum_{\rho\neq\sigma=1}^{d} \mathrm{tr}\, U_{y,\rho} U_{y+\rho,\sigma} U^{+}_{y+\sigma,\rho} U^{+}_{y,\sigma} \qquad (7)$$



where the product is over the link variables around an elementary square (plaquette) and the sum runs over all plaquettes on the lattice.

The Wilson loop amplitude is given by

$$W(C) = \frac{\prod_{y}\prod_{\rho} \int dU_{y,\rho}\, \mathrm{tr}(U_{x,\mu} U_{x+\mu,\nu} U_{x+\mu+\nu,\lambda} \cdots U_{x-\sigma,\sigma})\, e^{-\beta S}}{\prod_{y}\prod_{\rho} \int dU_{y,\rho}\, e^{-\beta S}}$$

$$= \langle \mathrm{tr}(U_{x,\mu} U_{x+\mu,\nu} U_{x+\mu+\nu,\lambda} \cdots U_{x-\sigma,\sigma}) \rangle \qquad (8)$$

for a contour C which connects lattice sites $x, x+\mu, x+\mu+\nu$, $x+\mu+\nu+\lambda,,, x-\sigma, x$ successively. $\beta = 1/g^2$ is the inverse temprature.

Now we reduce the model by identifying all link variables in the same direction

$$U_{y,\rho} \Rightarrow U_\rho \ .$$

The reduced model then has only d matrices $U_1, U_2,,, \cdot\cdot, U_d$ and its action is given by

$$S_r = - \sum_{\rho\neq\sigma=1}^{d} \mathrm{tr}(U_\rho U_\sigma U_\rho^+ U_\sigma^+) . \tag{9}$$

The analogue of the Wilson loop amplitude is defined by

$$W_r(C) = \frac{\prod_\rho \int dU_\rho \ \mathrm{tr}(U_\mu U_\nu U_\lambda \cdots U_\sigma) \ e^{-\beta S_r}}{\prod_\rho dU_\rho \ e^{-\beta S_r}}$$

$$= \langle \mathrm{tr}(U_\mu U_\nu U_\lambda \cdots U_\sigma) \rangle_r \tag{10}$$

where we have identified the contour C with the sequence of directions $(\mu, \nu, \lambda,,, \sigma)$

$$C \ ; \ (x, x+\mu, x+\mu+\nu, x+\mu+\nu+\lambda, \cdots, x-\sigma, x) \approx (\mu, \nu, \lambda, \cdots, \sigma) .$$

The above correspondence is one to one when we ignore over-all translations.

We remark that the reduced model Eq.(9) is ivariant under the phase transformation

$$U_\rho \rightarrow e^{i\theta} U_\rho \tag{11}$$

and this symmetry implies

$$W_r(C) = \langle \mathrm{tr}\, U_\mu U_\nu U_\lambda \cdots U_\sigma \rangle = 0 \tag{12}$$

for every open contour C. This is because in the case of an open contour there exists at least one direction ρ for which U_ρ and U_ρ^+ appear different number of times and W(C) has to vanish. It was pointed out[3] that if the symmetry Eq.(11) is left intact, i.e. not spontaneously broken, the equation of motion for the Wilson loop amplitudes in the original and reduced models become identical in the limit $N = \infty$ with g^2N fixes. Consequently the Wilson loop amplitudes agree

$$W(C) = W_r(C). \tag{13}$$

Thus the infinite-volume Wilson theory Eq.(7) and the one-site reduced model Eq.(9) become equivalent to each other in the large N limit.

Assumption of the preservation of the U(1) symmetry was checked by Monte Carlo computor simulation.[4] It was found that above a certain critical temprature $\lambda(=g^2N) = \lambda_c$ U(1) invariance

is unbroken, however, it is spontaneously broken below λ_c.
In the low temprature region the eigenvalues of the reduced link variables U_ρ are not uniformly distributed but concentrated around an arbitrary point on the unit circle and this spontaneously breaks U(1) symmetry. In this situation it is possible to restore the broken symmetry by integrating over the location of the concentration of the eigenvalues.
In the quenching procedure of refs. 4, 5 and 6 link variables are diagonalized as

$$U_\rho = V_\rho D_\rho V_\rho^+$$
$$(D_\rho)_{ij} = e^{i\theta_\rho^j} \delta_{ij} . \tag{14}$$

The angular variables θ_ρ^j's are held fixed when we first average over marices V_ρ's

$$W(C;\theta) = \frac{\int \prod_\rho dV_\rho \ \mathrm{tr}(V_\mu D_\mu V_\mu^+ V_\nu D_\nu V_\nu^+ \cdots V_\sigma D_\sigma V_\sigma^+) \ e^{-\beta S(V,\theta)}}{\int \prod_\rho dV_\rho \ e^{-\beta S(V,\theta)}} . \tag{15}$$

We then take an averaging over θ's

$$W_q(C) = \int d\mu(\theta) \ \ W(C;\theta) \tag{16}$$

with a suitable measure μ. It is now believed that the quenched reduced model agrees with the original Wilson theory for all temperature

$$W_q(C) = W(C). \tag{17}$$

Moreover when the quenched model is expanded into weak coupling series using the correspondence Eq.(6), it reproduces the planar perturbation theory of the continuum gauge fields.[6]

We thus have managed to reduce the problem of planar diagrams to the evaluation of a model with a limited number of unitary matrices. Hopefully we will be able to make further progress via the analysis of the model either by numerical or analytic techniques.

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