数理解析研究所講究録
471巻 1982年 168-182

Identities for divisor generating functions and their relations to a probability generating function

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1. Introduction

Identities for divisor generating functions are studied in e. g., [5], [9] and [10].

In [10] the following idntity is shown :

$$\sum_{n=1}^{\infty} nx^n \prod_{j=n+1}^{\infty} (1-x^j) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n(n+1)/2}}{(1-x)(1-x^2) \ \ldots \ (1-x^{n-1})(1-x^n)^2}$$

$$= \sum_{n=1}^{\infty} \sigma_0(n) x^n. \qquad (1.1)$$

In this paper we shall show identities for divisor generating functions $\sum_{n=1}^{\infty} \sigma_r(n)x^n$ which are generalization of (1.1) and their relation to a probability generating function. In the section 3, we shall apply this relation to an analysis of the data structure called a heap and evaluate the average value and the variance of the number of exchanges to insert a new element into a heap under certain assumptions.

2. Identities for divisor generating functions

In this section we shall show identities for divisor generating functions. In order to prove the identities we use the following standard abbreviations

$$(a)_n = \prod_{j=0}^{n-1} (1-aq^j),$$

$$(a)_\infty = \lim_{n\to\infty} (a)_n,$$

$$(a)_0 = 1.$$

We define functions M_m by

$$M_m = \sum_{n=1}^{\infty} n^m q^n (q^{n+1})_\infty .$$

We denote divisor generating functions $\sum_{n=1}^{\infty} \sigma_m(n) q^n$ by K_{m+1}. Namely $K_{m+1} = \sum_{n=1}^{\infty} \sigma_m(n) q^n$.

<u>Theorem 2.1.</u> For any $n \geqq 1$,

$$M_n = Y_n(K_1, \ldots , K_n),$$

where Y_n is the Bell polynomial defined by

$$Y_n(u_1, \ldots , u_n) = \sum_{\Pi(n)} \frac{n!}{k_1! \ldots k_n!} \left(\frac{u_1}{1!}\right)^{k_1} \ldots \left(\frac{u_n}{n!}\right)^{k_n},$$

where $\Pi(n)$ denotes a partition of n with

$$k_1 + 2k_2 + \ldots + nk_n = n.$$

Proof. Let

$$G(x, q) = (q)_\infty/(xq)_\infty. \qquad (2.1)$$

From Euler's partition formula ([7, p. 21])

$$\sum_{n=0}^{\infty} \frac{t^n}{(q)_n} = \frac{1}{(t)_\infty} \quad (\text{for } |q| < 1,\ |t| < 1), \qquad (2.2)$$

we get

$$G(x, q) = (q)_\infty \sum_{n=0}^{\infty} x^n q^n/(q)_n$$

$$= \sum_{n=0}^{\infty} x^n q^n (q^{n+1})_\infty. \qquad (2.3)$$

Putting $x = e^t$, we find

$$\frac{\partial^m}{\partial t^m} G(e^t, q)|_{t=0} = \sum_{n=1}^{\infty} n^m q^n (q^{n+1})_\infty = M_m.$$

Let $\log G(e^t, q) = h_1 t + h_2 t^2/2! + h_3 t^3/3! + \ldots$.

It follows from (2.1) that

$$\log G(e^t, q) = \log (q)_\infty - \log (e^t q)_\infty.$$

Clearly $-\log (1-e^t q) = \sum_{n=0}^{\infty} t^n/n! \sum_{j=1}^{\infty} j^{n-1} q^j$.

Therefore

$$-\log (e^t q)_\infty = -\sum_{j=1}^{\infty} \log (1-e^t q^j)$$

$$= \log (q)_\infty + \sum_{n=1}^{\infty} t^n/n! \sum_{j=1}^{\infty} j^{n-1} q^j/(1-q^j).$$

Thus $\quad h_{m+1} = \sum_{n=1}^{\infty} \sigma_m(n) q^n = K_{m+1}$ for any $m \geq 0$.

On the other hand

$$G(e^t, q) = \exp(K_1 t + K_2 t^2/2! + K_3 t^3/3! + \dots).$$

Hence $\quad \frac{\partial^n}{\partial t^n} G(e^t, q)\big|_{t=0} = Y_n(K_1, \dots, K_n)$.

This complete the proof of Theorem 2.1. □

We remark that $M_0 = (q)_\infty$.

By this theorem we have

$$M_1 = K_1, \tag{2.4}$$

$$M_2 = K_1^2 + K_2. \tag{2.5}$$

Next we shall study several variations of the identities.

A. From the q-analogue of Gauss's summation [1, p. 20]

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c/ab)^n}{(q)_n (c)_n} = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty}$$

$$(\text{for } |c| < |ab|, \ |q| < 1),$$

we get

$$G(x, q) = (q)_\infty/(xq)_\infty$$

$$= \lim_{\tau\to 0} \frac{(q)_\infty (x\tau)_\infty}{(xq)_\infty (\tau)_\infty}$$

$$= \lim_{\tau\to 0} \sum_{n=0}^{\infty} \frac{(x)_n (q/\tau)_n \tau^n}{(q)_n (xq)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(x)_n (-1)^n q^{n(n+1)/2}}{(q)_n (xq)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n} \times \frac{1-x}{1-xq^n} . \qquad (2.6)$$

Thus

$$\frac{\partial^m}{\partial x^m} G(x, q) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(q)_{n-1}} \times \frac{m!\, q^{(m-1)n}}{(1-xq^n)^{m+1}} . \qquad (2.7)$$

From (2.7) we have identities for divisor generating functions. For instance,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(q)_{n-1}} \times \frac{2q^n}{(1-q^n)^3} = K_2 - K_1 + K_1^2,$$

since

$$K_2 = e^{2t} G''(e^t)/G(e^t) + e^t G'(e^t)/G(e^t) - e^{2t} G'(e^t)^2/G(e^t)^2 \big|_{t=0}, \qquad (2.8)$$

where $G^{(j)}(x) = \frac{\partial^j}{\partial x^j} G(x, q), \qquad j = 1, 2$.

We remark that (1.1) is obtained from (2.4) and (2.7).

B. Let

$$H(x,q) = (xq)_\infty/(q)_\infty.$$

Then

$$\log H(e^t,q) = \sum_{n=1}^{\infty} (-K_n)t^n/n!.$$

On the other hand, from Cor. 2.2 [1, p. 19], we have

$$(xq)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n x^n q^{n(n+1)/2}}{(q)_n}.$$

Therefore we have

$$\frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n n^m q^{n(n+1)/2}}{(q)_n} = Y_m(-K_1, \ldots, -K_m),$$

for $m \geqq 1$.

We now note that $G(1, q) = 1$. Then we can regard the function $\sum_{n=0}^{\infty} x^n q^n (q^{n+1})_\infty$ as a probability generating function in which $P(X = n)$ is defind to be $q^n(q^{n+1})_\infty$, for $0 < q < 1$. Let $\overline{G}(x, q) = \sum_{n=0}^{\infty} x^n q^n (q^{n+1})_\infty$.

Then the radius of convergence of $\overline{G}(x, q)$ is $1/q$. Therefore all monents exist and the exponential moment generating function $\sum_{r=0}^{\infty} M_r t^r/r!$ converges for $|t| < \log(1/q)$ and equals $\overline{G}(e^t, q)$ ([4, p. 285]). We note that M_r is the r-th monent.

On the other hand, h_r is the r-th cumulant for any $r \geqq 1$, and so is K_r. Therefore K_1 and K_2 are the average value and the variance respectively.

3. Applications to an analysis of heaps

We now give a combinatorial interpretation of the probability generating function. We shall study an analysis of the data structure called a heap and apply the relation between the identities and the probability generating function to the analysis.

A heap is defined to be a t-ary labelled tree $(t \geqq 2)$ such that the element associated with each vertex is smaller than the elements associated with its sons.

Our aim is to evaluate the average number of exchanges required to insert an element into a heap. We also evaluate its variance.

We consider heaps on complete t-ary trees with t^n-1 vertices (See [6,p.401]). We call an element to be inserted into a heap an input element. Without loss of generality we may assume that a set of elements on a t-ary tree and an input element is equal to a set $\{1, 2, \ldots, t^n\}$. We consider a set of all pairs of a heap and an input element which satisfy the above conditions. We denote the set of pairs by H_n. An input element is inserted into the heap at the shallowest and leftmost empty position in the tree. The position is called the input position.

Fig. 3.1.

We assume that each pair of a heap and an input element is equally likely (See [7,p.155]).

Let P_j^n be the probability that an input element is to be exchanged exactly j times in a heap of t^n-1 elements when the input element is inserted at the input position.
In the case depicted in Fig. 3.1 the input element is exchanged once.

Proposition 3.1. For any positive integer n,

$$P_j^n = (1-1/t^n)P_j^{n-1}, \quad \text{if } 0 \leq j < n,$$

$$P_n^n = 1/t^n,$$

$$P_j^n = 0, \quad \text{if } n < j.$$

Proof. It is obvious that $P_j^n = 0$ if $n < j$.
Since an input is exchanged n times in a heap of t^n-1 elements if and only if the input element is equal to 1, it follows from the above assumption that $P_n^n = 1/t^n$.

We now note the following property.
Let S be the subtree of a complete t-ary tree T with t^n vertices that contains a son of the root of T as the root as well as the rightmost of the deepest elements in T.
Then S is also a complete t-ary tree with t^{n-1} vertices.

In the case that $0 \leq j < n$, an input element is not equal to 1. Let U_n be a subset of H_n consisting of all pairs of a heap and an input element which is not equal to 1. It is clear that for any pair in U_n, an input element is exchanged in the subtree mentioned above. We denote the set of pairs of a subheap on such a subtree and an input element by D_{n-1}.

We now define a mapping ϕ from U_n onto H_{n-1} as follows. For any pair x in U_n, we can find a pair y in D_{n-1} naturally. Let the set of elements of the heap and the input element in y be $\{i(1), \ldots, i(t^{n-1})\}$, where $i(j) < i(k)$ if $j < k$. Substituting an integer j for each element i(j) in y, we get a pair z in H_{n-1}. Then we define the mapping ϕ by $\phi(x) = z$.

It can be easily observed that $|\phi^{-1}(z_1)| = |\phi^{-1}(z_2)|$, for any z_1, z_2 in H_{n-1}, where $|S|$ denotes the cardinality of a set S. Let $|\phi^{-1}(z)| = c$ for $z \in H_{n-1}$. Let d_j be the number of pairs in H_{n-1} in which the input element is exchanged exactly j times in the heap. Then we have

$$P_j^{\,n} = d_j \times c/|H_n| = (1-1/t^n)P_j^{\,n-1}.$$

This completes the proof of Proposition 3.1. □

We consider the probability generating function $\sum_{j=0}^{\infty} P_j{}^n x^j$.

Putting $1/t = q$ in the function, we have a function

$$G_n(x, q) = \sum_{k=0}^{n} x^k q^k \prod_{j=k+1}^{n} (1-q^j).$$

Extending this function we now obtain a function

$$\bar{G}(x, q) = \sum_{k=0}^{\infty} x^k q^k (q^{k+1})_\infty,$$

which has already appeared in the previous section. We recall that this function is a probability generating function and that K_1 and K_2 are the average value and the variance, respectively.

Let $A_n(q) = \frac{\partial}{\partial x} G_n(x, q)|_{x=1}$ and

$$V_n(q) = \frac{\partial^2}{\partial x^2} G_n(x, q)|_{x=1} + A_n(q) - A_n(q)^2.$$

It can be easily proved that

$$\log G_n(e^t, q) = A_n(q)t + V_n(q)t^2/2! + \ldots .$$

Then $A_n(1/t)$ is the average number of exchanges for updating an input element in a pair in H_n and $V_n(1/t)$ is equal to its variance.

Let α be a positive number such that $(1+\alpha)q < 1$. Then we have

Proposition 3.2. There exist a positive integer N and positive numbers C_1 and C_2 such that

$$|A_n(q) - K_1(q)| \leqq C_1((1+\alpha)q)^n \qquad \text{and}$$

$$|V_n(q) - K_2(q)| \leqq C_2((1+\alpha)q)^n, \qquad \text{for any } n \geqq N.$$

Proof. It is clear that

$$A_n(q) = \sum_{k=1}^{n} kq^k \prod_{j=k+1}^{n} (1-q^j).$$

Let $$A_n(q) = \sum_{k=1}^{n(n+1)/2} a_k q^k.$$

We recall that $K_1 = \sum_{k=1}^{\infty} \sigma_0(k) q^k = \sum_{k=1}^{\infty} kq^k (q^{k+1})_\infty$.

Clearly $a_k = \sigma_0(k)$ if $k \leqq n$, ([10, Theorem 1]). (3.1)

Now we consider a function

$$B(q) = \sum_{k=1}^{\infty} kq^k \prod_{j=1}^{\infty} (1+q^j).$$

Let $$B(q) = \sum_{k=1}^{\infty} b_k q^k.$$

Then $|a_k| \leqq |b_k|$ for any k. (3.2)

It is clear that the radius of convergence of $B(q)$ is equal to 1. It is known [5, p. 260] that

$O(\sigma_0(k)) = k^\delta$ for all positive δ. Therefore for the positive number α there exists a positive integer N such that

$$|b_k| \leqq (1+\alpha)^k, \qquad \sigma_0(k) \leqq (1+\alpha)^k \qquad \text{for any } k \geqq N.$$

Hence it follows from (3.1) and (3.2) that

$$|A_n(q) - K_1(q)| \leqq C_1((1+\alpha)q)^n .$$

By the similar way we get the second inequality since we have from [5, p. 266],

$$\sigma_1(k) = O(n^{1+\delta}) \qquad \text{for all} \quad \text{pôsitive } \delta,$$

and from (2.4) and (2.8),

$$K_2 = G''(1, q) + K_1 - K_1^2 . \quad \square$$

We now consider the higher monents.
Let $M_{r,n}$ be the r-th monent defined by $G_n(x, q)$. Then

$$M_{r,n} = \sum_{k=1}^{n} k^r q^k \prod_{j=k+1}^{n} (1-q^j).$$

On the other hand, the r-th monent M_r defined by $\overline{G}(x, q)$ is

$$\sum_{k=1}^{\infty} k^r q^k \prod_{j=k+1}^{\infty} (1-q^j).$$

Thus there exist α, N and C such that $(1+\alpha)q < 1$, and

$$|M_r - M_{r,n}| \leqq C((1+\alpha)q)^n , \qquad \text{for } n \geqq N .$$

We note that the r-th monents $M_{r,n}$ satisfy the following recurrence

$$M_{r,n} = n^r q^n + (1-q^n)M_{r,n-1} .$$

Therefore the sequence $\{M_{r,n}\}$ is monotone increasing.

Theorem 2.1 tells us that

$$M_r = Y_r(K_1, \ldots, K_r).$$

In order to evaluate the monents M_r, it suffices to evaluate the cumulants K_j. For the purpose, we use the formula on Lambert series.

$$\sum_{n=1}^{\infty} a_n x^n/(1-x^n) = \sum_{n=1}^{\infty} x^{n^2}(a_n + \sum_{k=1}^{\infty}(a_n+a_{n+k})x^{kn}).$$

Then we can compute the values $K_j(q)$ rapidly.
The values $K_1(q)$ and $K_2(q)$ for $q = 1/2, 1/3, 1/4, 1/5,$ are shown in the following table.

Table 3.1

Remarks When $t = 2$, Proposition 3.1 is essentially the same as Theorem 1 in [8] and also as Proposition in [3] .
But it is well known that heaps on t-ary trees $(t \geq 3)$ are useful data structures. So we note the proposition.

Doberkat shows in [3] that

$$M_r \sim 2r![(\log 2)^{-r-1} + 2\mathrm{Re}(2\pi i)^{-r-1}\zeta(r+1, 1-i\log 2/(2\pi))],$$

where $\zeta(n,a) = \sum_{k\geq 0}(k+a)^{-n}$,

as $r \to \infty$.

And he expects the readers to have estimates for $M_{r,n}$ in [3].
In this paper we give an answer to his problem.

Acknowledgement

The author thanks Professor G. E. Andrews for showing the author the identity (2.6) in [2].

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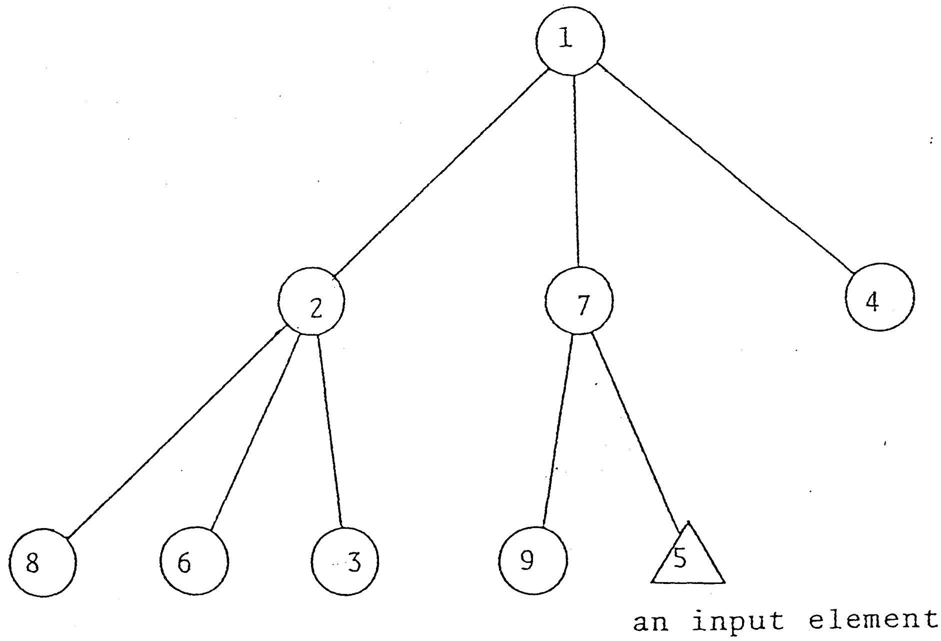

Fig. 3.1

Average values and variances		
q	$K_1(q)$	$K_2(q)$
1/2	1.60669	2.74403
1/3	0.68215	0.94943
1/4	0.42109	0.53692
1/5	0.30173	0.36603

Table 3.1