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Difference Analogue of Nonlinear Evolution Equations in Hamiltonian Form

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The followings are only the summary of the results on the discretization of Hamilton's equation of motion. Details will be published elsewhere.

We define the following basic difference operators with the increment ε. The central difference operator;

$$\Delta_x f(x) = (2/\varepsilon)\sinh\left(\frac{\varepsilon}{2}\frac{\partial}{\partial x}\right) f(x) \tag{1}$$

$$= \varepsilon^{-1}[f(x+\varepsilon/2) - f(x-\varepsilon/2)] . \tag{2}$$

The averaging operator;

$$\Pi_x f(x) = \cosh\left(\frac{\varepsilon}{2}\frac{\partial}{\partial x}\right) f(x) \tag{3}$$

$$= [f(x+\varepsilon/2) + f(x-\varepsilon/2)]/2 . \tag{4}$$

Note that $f(x \pm \varepsilon/2) = \Pi_x f(x) \pm (\varepsilon/2)\,\Delta_x f(x)$. (5)

The binary operators;

$$\Delta_x f(x)\cdot g(x) = (2/\varepsilon)\ \sinh\ (\tfrac{\varepsilon}{2} D_x)\ f(x)\cdot g(x) \tag{6}$$

$$= \varepsilon^{-1}[f(x + \varepsilon/2)g(x - \varepsilon/2) - f(x - \varepsilon/2)g(x + \varepsilon/2)], \tag{7}$$

$$\Pi_x f(x)\cdot g(x) = \cosh\ (\tfrac{\varepsilon}{2} D_x)\ f(x)\cdot g(x) \tag{8}$$

$$= [f(x + \varepsilon/2)g(x - \varepsilon/2) + f(x - \varepsilon/2)g(x + \varepsilon/2)]/2\ . \tag{9}$$

The difference operators with double increment 2ε ;

$$\Delta_{2x} f(x) = \Delta_x \Pi_x f(x) \tag{10}$$

$$= (2\varepsilon)^{-1}[f(x + \varepsilon) - f(x - \varepsilon)]\ , \tag{11}$$

$$\Pi_{2x} f(x) = (\Pi_x{}^2 + (\varepsilon/2)^2 \Delta_x{}^2)\ f(x) \tag{12}$$

$$= [f(x + \varepsilon) + f(x - \varepsilon)]/2\ . \tag{13}$$

Note that $\Pi_x{}^2 - (\varepsilon/2)^2\Delta_x{}^2 = 1$. (14)

With these difference operators, we find the following formulae;

$$[\Delta_{2x} f(x)]g(x) + f(x)[\Delta_{2x} g(x)] = \Delta_x[\Pi_x f(x)\cdot g(x)]\ , \tag{15}$$

$$[\Delta_x{}^2 f(x)]g(x) - f(x)[\Delta_x{}^2 g(x)] = \Delta_x[\Delta_x f(x)\cdot g(x)]\ . \tag{16}$$

We have the chain rule of differentiation

$$\frac{d}{dx} f[g(x)] = [\frac{df}{dg}][\frac{dg}{dx}] \tag{17}$$

A difference analogue of the chain rule is stated as follows

$$\Delta_x f[g(x)] = [\Delta_g f\Big|_x][\Delta_x g(x)] , \tag{18}$$

$$\Pi_x f[g(x)] = \Pi_g f(g)\Big|_x , \tag{19}$$

where

$$\Delta_g f(g)\Big|_x = (2/\varepsilon')[f(g' + \varepsilon'/2) - f(g' - \varepsilon'/2)] \Bigg| \begin{array}{l} g' = \Pi_x g(x) \\ \varepsilon' = \varepsilon\Delta_x g(x) . \end{array} \tag{20}$$

$$\Pi_g f(g)\Big|_x = [f(g' + \varepsilon'/2) - f(g' - \varepsilon'/2)]/2 \Bigg| \begin{array}{l} g' = \Pi_x g(x) \\ \varepsilon' = \varepsilon\Delta_x g(x) . \end{array} \tag{21}$$

The concept of total and partial differential is crucial to write the equation of motion in Hamiltonian form. When f is a function of **functions** $g_1, g_2, \cdots, g_n$, we have

$$\frac{d}{dx} f[g_1, g_2, \cdots, g_n] = \sum_{i=1}^{n} [\frac{\partial f}{\partial g_i}][\frac{dg_i}{dt}] . \tag{22}$$

A difference analogue of eq.(22) is expressed **with the form**

$$\Delta_x f[g_1, g_2, \cdots, g_n] = \sum_{i=1}^{n} [\Delta_{g_i} f\Big]_x][\Delta_x g_i] , \tag{23}$$

where $\Delta_{g_i} f\Big]_x$ is expressed with rather complicated forms in general cases, but for n = 2, we find a simpler one

$$\Delta_{g_1} f\Big]_x = \Delta_{g_1} \Pi_{g_2} f(g_1, g_2)\Big|_x , \tag{24}$$

$$\Delta_{g_2} f\Big]_x = \Pi_{g_1} \Delta_{g_2} f(g_1, g_2)\Big|_x . \tag{25}$$

Now we consider the nonlinear evolution equations in Hamiltonian form. Let H be a functional of (ρ, u) and their x-derivatives

$$H = \int_{-\infty}^{\infty} \mathcal{H}(\rho(x, t), u(x, t))\, dx . \tag{26}$$

Hereafter we limit our investigations only to the nonlinear evolution equations in Hamiltonian form which can be written in the form

$$\frac{du}{dt} = \frac{\delta \mathcal{H}}{\delta \rho}, \qquad \frac{d\rho}{dt} = -\frac{\delta \mathcal{H}}{\delta u}, \text{ or } \begin{pmatrix} \frac{du}{dt} \\ \\ \frac{d\rho}{dt} \end{pmatrix} = Q \begin{pmatrix} \frac{\delta \mathcal{H}}{\delta u} \\ \\ \frac{\delta \mathcal{H}}{\delta \rho} \end{pmatrix} . \tag{27}$$

If H does not involve ρ explicitely, the Hamiltonian form reduces to

$$\frac{du}{dt} = Q \frac{\delta \mathcal{H}}{\delta u}, \tag{28}$$

where $\mathcal{H}$ is a Hamiltonian density, $\frac{\delta}{\delta u}$ is the Euler derivative with respect to u and Q is a skew-symmetric operator defined by

$$\int_{-\infty}^{\infty} A(QB)dx = -\int_{-\infty}^{\infty} B(QA)dx \tag{29}$$

One of the advantages of writing an evolution equation in Hamiltonian form is that it ensures that H is the conserved quantity;

$$\frac{dH}{dt} = \frac{d}{dt} \int \mathcal{H}(\rho,\ u)dx \tag{30}$$

$$= \int \left(\frac{\delta \mathcal{H}}{\delta \rho}\frac{d\rho}{dt} + \frac{\delta \mathcal{H}}{\delta u}\frac{du}{dt}\right)dx \tag{31}$$

which vanishes due to eq.(27). If H does not involve ρ explicitely, we have

$$\frac{dH}{dt} = \int \frac{\delta \mathcal{H}}{\delta u}\frac{du}{dt}\, dx \tag{32}$$

$$= \int \frac{\delta \mathcal{H}}{\delta u} \left(Q\, \frac{\delta \mathcal{H}}{\delta u}\right)dx \tag{33}$$

which vanishes because of the skew-symmetric preperty of Q.

The difference analogue of the Euler operator is obtained by using eq. (23), the total difference expressed with the partial differences and eq. (15), the basic relation leading to the difference analogue of the integration by parts.

Here we define the central difference operator with respect to time by

$$\Delta_t u(t) = \delta^{-1}[u(t + \delta/2) - u(t - \delta/2)]\ . \tag{34}$$

We write the difference analogue of eqs.(27) and (28) in the form

$$\Delta_t u = \left.\frac{\delta \mathcal{H}}{\delta u}\right|_t , \qquad \Delta_t \rho = - \left.\frac{\delta \mathcal{H}}{\delta u}\right|_t , \tag{35}$$

$$\Delta_t u = Q \left.\frac{\delta \mathcal{H}}{\delta u}\right|_t \tag{36}$$

where the r, h, s is obtained by using eqs.(23) and (15), whose explicit form for the given H will be found in the following list.

1. Evolution equation

$$u_{xx} - u_{tt} = \frac{\partial}{\partial u} U(u) \tag{37}$$

(Nonlinear Klein-Gordon eq.).

Hamilton density

$$\mathcal{H} = \frac{1}{2} \rho^2 + \frac{1}{2}(\Delta_{2x} u)^2 + U(u) \ . \tag{38}$$

Euler Derivative

$$\left.\frac{\delta \mathcal{H}}{\delta \rho}\right|_t = \Pi_t \rho \ , \tag{39}$$

$$\left.\frac{\delta \mathcal{H}}{\delta u}\right|_t = \Delta_{2x}{}^2 \Pi_t \rho - \Delta_u U(u)\Big|_t \ . \tag{40}$$

Difference analogue of the evolution equation

$$\Delta_t \rho = - \left.\frac{\delta \mathcal{H}}{\delta u}\right|_t \ , \tag{41}$$

$$\Delta_t u = \left.\frac{\delta \mathcal{H}}{\delta \rho}\right|_t \ . \tag{42}$$

2. Evolution equation

$$u_t = 6uu_x + u_{xxx} \tag{43}$$

(KdV equation) .

(i) Hamiltonian density

$$= u^3 - \frac{1}{2}(\Delta_{2x}u)^2 , \qquad (44)$$

Euler derivative

$$\left.\frac{\delta \mathcal{H}}{\delta u}\right|_t = 3(\Pi_t u)^2 + (\delta\Delta_t u/2)^2 + (\Delta_{2x}{}^2\Pi_t u) . \qquad (45)$$

Difference analogue of the evolution equation

$$\Delta_t u = \Delta_{2x} \left.\frac{\delta \mathcal{H}}{\delta u}\right|_t . \qquad (46)$$

(ii) Hamiltonian density

$$= \frac{1}{2} u^2 . \qquad (47)$$

Euler derivative

$$\left.\frac{\delta \mathcal{H}}{\delta u}\right|_t = \Pi_t u . \qquad (48)$$

Difference analogue of the evolution **equation**

$$\Delta_t u = [\Delta_{2x}{}^3 + 2(\Pi_t u)\Delta_{2x} + 2\Delta_{2x}(\Pi_t u)] \left.\frac{\delta \mathcal{H}}{\delta u}\right|_t . \qquad (49)$$

3. Evolution equation

$$u_t = 10u^3 u_x + u_{xxx} . \tag{50}$$

Hamiltonian density

$$\mathcal{H} = \frac{1}{2} u^2 . \tag{51}$$

Difference analogue of the evolution equation

$$\Delta_t u = [\Delta_{2x}{}^3 + (\Pi_t u)^3 \Delta_{2x} + \Delta_{2x}(\Pi_t u)^3 + (\Pi_t u)^2 \Delta_{2x}(\Pi_t u) + (\Pi_t u)\Delta_{2x}(\Pi_t u)^2](\Pi_t u) . \tag{52}$$

The skew-symmetric operators in eqs.(49) and (52) are constructed so as to conserve the quantity Σu as well as the Hamiltonian $\Sigma\mathcal{H}$.