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Recent Topics on Symmetries and Hidden Symmetries of Nonlinear Fields

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1. Symmetries of Gravitational Fields

1.1 Introduction

The stationary axially symmetric vacuum gravitational fields are interpreted as chiral fields over 2-dimensional space [1]. Thus, it is important to consider the internal symmetries of field equations. Note especially the works of Kinnersley and Chitre [2]. They have exponentiated the infinitesimal transformations which are due to the internal symmetries.

On the other hand, the stationary axially symmetric vacuum Einstein equations are equivalent to the static axially symmetric SU(2) self-dual gauge field equations which admit monopole solutions [3]. This suggests that we may apply the methods developed in general relativity to the problem of finding classical solutions of gauge fields [4]. A similar relationship is known between the finite Toda lattice equations and the spherically symmetric self-dual equations [5]. However, it is not obvious whether the formalism generating multi-instanton solutions such as the Atiyah-Ward construction [6] is applicable to lower dimensional cases.

Let us attend to the result of Corrigan, Fairlie, Yates and Goddard [7]. They have found a Bäcklund transformation for SU(2) self-dual gauge fields and have succeeded in integrating the ansatz of Atiyah and Ward.

In this chapter, we discuss the internal symmetries of the stationary axially symmetric gravitational fields and present a Bäcklund transformation. This nontrivial transformation is similar to that of Corrigan et al. and is a special case of the Kinnersley-Chitre transformation of the Geroch group. For the initial ansatz of field equations, we show that the Bäcklund transformation can be integrated and generates the hierarchy of ansatz.

1.2 Stationary Axially Symmetric Einstein Equations

It is known that the stationary axially symmetric vacuum Einstein equations reduce to

$$\partial_\rho(\rho\partial_\rho Q\cdot Q^{-1}) + \partial_z(\rho\partial_z Q\cdot Q^{-1}) = 0, \tag{1}$$

where $Q = Q(\rho, z)$ is a real symmetric 2×2 matrix with the supplementary condition $\det Q = -\rho^2$ [1]. Papapetrou's parametrization $Q = (q_{\mu\nu})$, $q_{11} = f$, $q_{12} = q_{21} = f\omega$, $q_{22} = f^2\omega^2 - \rho^2 f^{-1}$, implies that there exists a potential $\psi = \psi(\rho, z)$ defined by

$$\partial_\rho\psi = \rho^{-1}f^2\partial_z\omega, \quad \partial_z\psi = -\rho^{-1}f^2\partial_\rho\omega. \tag{2}$$

Then, we obtain the field equations

$$\begin{aligned} &f(\partial_\rho^2+\rho^{-1}\partial_\rho+\partial_z^2)f - (\partial_\rho f)^2 - (\partial_z f)^2 + (\partial_\rho\psi)^2 + (\partial_z\psi)^2 = 0, \\ &\partial_\rho(\rho f^{-2}\partial_\rho\psi) + \partial_z(\rho f^{-2}\partial_z\psi) = 0. \end{aligned} \tag{3}$$

It must be noted that a discrete mapping

$$I: (f, \omega) \longrightarrow (\rho f^{-1}, i\psi) \tag{4}$$

transforms (1) to (3) directly. Let P be 2×2 matrix as $P = (p_{\mu\nu})$

$p_{11} = f^{-1}$, $p_{12} = p_{21} = f^{-1}\psi$, $p_{22} = f^{-1}(f^2 + \psi^2)$. We can prove that the field equations (3) are equivalent to

$$\partial_\rho(\rho\partial_\rho P \cdot P^{-1}) + \partial_z(\rho\partial_z P \cdot P^{-1}) = 0. \tag{5}$$

1.3 Internal Symmetries and Bäcklund Transformation

Let H be an SL(2,$\mathbb{R}$) constant matrix. The equation (5) is invarient under a rotation

$$H: P \longrightarrow HPH^{tr}. \tag{6}$$

The invarience group defined by (6) is a subgroup of the Geroch group. Particularly, we set $H = (\varepsilon_{\mu\nu})$ which is an appropriate linear combination of generators for gauge and Ehlers transformations. Providing that $f^2 + \psi^2 \neq 0$, we have

Lemma γ If (f, ψ) satisfies (3), so does (f', ψ') defined by

$$f' = f(f^2 + \psi^2)^{-1}, \quad \psi' = -\psi(f^2 + \psi^2)^{-1}. \tag{7}$$

Next, we consider another internal symmetry of (3). Combining the definition (2) and the mapping (4), we can prove

Lemma β Let (f, ψ) be a solution of (3), then (f', ψ') defined by

$$f' = \rho f^{-1}, \quad \partial_\rho\psi' = -i\rho f^{-2}\partial_z\psi, \quad \partial_z\psi' = i\rho f^{-2}\partial_\rho\psi \tag{8}$$

is also a solution.

Though the transformations β and γ are discrete, a product transformation α: $\alpha = \beta \circ \gamma$ has a nontrivial effect. The operation of α, a Bäcklund transformation, is given by the following proposition.

<u>Proposition α</u> The transformation α acts on an initial sulution (f, ψ) of (3) according to

$$
\begin{aligned}
f' &= \rho f^{-1}(f^2 + \psi^2), \\
\partial_\rho \psi' &= i\rho f^{-2}(f^2 + \psi^2)^2 \cdot \partial_z\{\psi(f^2 + \psi^2)^{-1}\}, \qquad (9)\\
\partial_z \psi' &= -\, i\rho f^{-2}(f^2 + \psi^2)^2 \cdot \partial_\rho\{\psi(f^2 + \psi^2)^{-1}\},
\end{aligned}
$$

where (f', ψ') is an another sulution of (3).

We remark that if one operates α even times, one can derive real potentials from real ones.

<u>1.4</u> <u>Hierarchy of Ansatz and Concrete Solutions</u>

We introduce new ansatz P_2 defined by Δ as follows.

<u>Proposition Δ</u> A solution of (3) is given by $f = \partial_\rho \Delta$, $\psi = \partial_z \Delta$, where $\Delta = \Delta(\rho, z)$ is a solution of

$$(\partial_\rho{}^2 - \rho^{-1}\partial_\rho + \partial_z^2)\Delta = 0. \qquad (10)$$

This heuristic consideration leads us to seek hierarchy of P_2 by means of the Bäcklund transformation α and the internal symmetries β and γ.

Let us summarize the results. We prepare the variables Δ_r, $r = 0, 1, \cdots$,

$$\partial_\rho \Delta_r = -\rho \partial_z \Delta_{r-1},$$
$$\partial_z \Delta_r = \rho \partial_\rho \Delta_{r-1} + 2(1-r)\Delta_{r-1}, \tag{11}$$

which lead to $\{\partial_\rho^2 + (1-2r)\rho^{-1}\partial_\rho + \partial_z^2\}\Delta_r = 0$. Using $\{\Delta_r\}$, we can derive hierarchy of the ansatz $P_{\ell+1}$ and P'_ℓ, $\ell = 1, 2, \cdots$. The relationship between $P_{\ell+1}$ and P'_ℓ is shown in the following diagram;

$$\begin{tikzcd}
 & P_2 \arrow[r, "\alpha"] \arrow[d, leftrightarrow, "\gamma"] \arrow[dl, leftrightarrow, "\beta"'] & P_3 \arrow[r, "\alpha"] \arrow[d, leftrightarrow, "\gamma"] \arrow[dl, leftrightarrow, "\beta"'] & {} \arrow[dl, "\beta"] \\
P'_1 & P'_2 \arrow[l, "\alpha"] & P'_3 \arrow[l, "\alpha"] & {} \arrow[l, "\alpha"]
\end{tikzcd}$$

Each ansatz is integrated to be characterized by determinants of $\{\Delta_r\}$. The first five ansatz are written as follows.

$$P'_1:\ (f, \psi) = \left(\frac{1}{\Delta_0}, \frac{-i}{\Delta_0}\right),$$

$$P_2:\ (f, \psi) = (\rho\Delta_0, \Delta_1),$$

$$P'_2:\ (f, \psi) = \left(\frac{\rho\Delta_0}{\begin{vmatrix} \rho\Delta_0 & \Delta_1 \\ -\Delta_1 & \rho\Delta_0 \end{vmatrix}}, \frac{\Delta_1}{\begin{vmatrix} \rho\Delta_0 & \Delta_1 \\ -\Delta_1 & \rho\Delta_0 \end{vmatrix}} \right),$$

$$P_3:\ (f, \psi) = \left(\frac{\begin{vmatrix} \rho\Delta_0 & \Delta_1 \\ -\Delta_1 & \rho\Delta_0 \end{vmatrix}}{\Delta_0}, \frac{i\begin{vmatrix} \Delta_1 & \Delta_2 \\ -\Delta_0 & \rho\Delta_1 \end{vmatrix}}{\Delta_0} \right),$$

$$P_3': (f, \psi) = \left(\frac{\begin{vmatrix} \rho\Delta_0 & \Delta_1 \\ -\Delta_1 & \rho\Delta_0 \end{vmatrix}}{\begin{vmatrix} \rho^2\Delta_0 & \Delta_1 & -\Delta_2 \\ -\Delta_1 & \Delta_0 & \Delta_1 \\ -\Delta_2 & -\Delta_1 & \rho^2\Delta_0 \end{vmatrix}}, \frac{-i\begin{vmatrix} \Delta_1 & \Delta_2 \\ -\Delta_0 & \Delta_1 \end{vmatrix}}{\begin{vmatrix} \rho^2\Delta_0 & \Delta_1 & -\Delta_2 \\ -\Delta_1 & \Delta_0 & \Delta_1 \\ -\Delta_2 & -\Delta_1 & \rho^2\Delta_0 \end{vmatrix}} \right).$$

We notice that P_ℓ and P_ℓ' (ℓ: odd) give real metric coefficients via the discrete mapping I.

Finally, we give concrete solutions. For the ansatz P_1', a special solution $\Delta_0 = R^{-1}$, $R = \{\rho^2 + (z-a)^2\}^{1/2}$, $a \in \mathbb{R}$, derives a metric

$$ds^2 = -\rho^{-1/2}(d\rho^2 + dz^2) + \rho R^{-1}dt^2 - 2\rho dt d\phi.$$

The variable Δ_0 leads to Δ_1 and Δ_2 through (11),

$$\Delta_1 = \{bR - (z-a)\}R^{-1},$$

$$\Delta_2 = \{R^2 - 2b(z-c)R + (z-a)^2\}R^{-1},$$

$a, b, c \in \mathbb{R}$. Then these variables give the metric coefficients

$$f = \rho\{(b^2+1)R - 2b(z-a)\}^{-1},$$

$$\omega = \{(b^2+1)\rho^2 + (b^2-1)(z-a)^2 + 2b(2z-a-c)R\}R^{-1},$$

of the ansatz P_3. The ansatz P_3' is also described by the above variables.

2. Hidden Symmetries of Chiral Fields

2.1 Infinitesimal Transformations

Recently, several works have been done for the hidden symmetries of chiral fields. Dolan and Roos [8] have proposed an infinitesimal transformation which shifts the Lagrangian density by a total divergence without the equations of motion. Refs.[9] have proved the same results systematically by using a generating function.

Let us first review Refs. [8,9]. The equations of motion for chiral fields are

$$\partial_\mu A_\mu = 0, \quad A_\mu = g^{-1}\partial_\mu g, \tag{12}$$

where $g = g(x)$ is an element of Lie group $\underline{G}$. The infinitesimal transformations are defined by

$$g \longrightarrow g + \delta_v^{(n)} g, \quad \delta_v^{(n)} g = - g\lambda^{(n)}, \tag{13}$$

$n = 1, 2, \cdots$. Here $\mathbf{v}$ is an element of Lie algebra $\mathcal{G}$ and $\lambda^{(n)}$ are given by $\partial_1\lambda^{(n+1)} = \partial_0\lambda^{(n)} + [A_0, \lambda^{(n)}]$, $\lambda^{(0)} = v$, recursively. Introducing a generating function $S(\zeta)$: $S(\zeta) = \sum_{n=0}^{\infty} \lambda^{(n)}\zeta^n$, $\zeta \in \mathbb{C}$ and a matrix function $Y(\zeta)$: $S(\zeta) = Y(\zeta)vY(\zeta)^{-1}$, one has

$$(\partial_1 - \zeta\partial_0)Y(\zeta) = \zeta A_0 Y(\zeta). \tag{14}$$

In this section we translate the infinitesimal transformations (13) into the language of the linear problem (14). We assume that there exists a fundamental solution matrix $Y(\zeta)$, holomorphic near the origin $\zeta = 0$, $Y(0) = 1$ and $\det Y(\zeta) = 1$. Since (13) induces an infinitesimal transformation on the potential A_0;

$$A_0 \longrightarrow A_0 - \delta_v^{(n)} A_0, \quad \delta_v^{(n)} A_0 = \partial_0 \lambda^{(n)} + [A_0, \lambda^{(n)}], \quad (15)$$

$Y(\zeta)$ changes to $Y(\zeta) - Z^{(n)}(\zeta)Y(\zeta)$, where

$$(\partial_1 - \zeta\partial_0)Z^{(n)}(\zeta) = \zeta\{\partial_0\lambda^{(n)} + [A_0, \lambda^{(n)} + Z^{(n)}(\zeta)]\}. \quad (16)$$

2.2 Riemann-Hilbert Transformations

The authors [10] have constructed the Riemann-Hilbert transformations for chiral fields and the anti-self-dual gauge fields. We will show that the infinitesimal transformations (15) are essentially deduced from the Riemann-Hilbert transformations. For the simplicity, we restrict ourselves to the case of $\underline{G} = SL(n,\mathbb{C})$.

Consider the Riemann-Hilbert problem as follows.

$$\begin{aligned} &X_-(\zeta') = X_+(\zeta')H(\zeta') \quad \text{for } \zeta' \in C, \\ &H(\zeta) = Y(\zeta)u(\zeta)Y(\zeta)^{-1}, \quad X_+(0) = 1. \end{aligned} \quad (17)$$

Here C is a small circle with the center at $\zeta = 0$ such that $Y(\zeta)$ is holomorphic in $C \cup C_+$, C_+ (C_-) denotes the inside (outside) of C. The matrix $u(\zeta)$ is analytic in C and belongs to $SL(n,\mathbb{C})$. Then it can be proved that $\tilde{Y}(\zeta)$ and $\tilde{A}_0$ introduced by

$$\begin{aligned} &\tilde{Y}(\zeta) = X_+(\zeta)Y(\zeta) \text{ in } C_+, \quad X_-(\zeta)Y(\zeta)u(\zeta)^{-1}, \text{ in } C_-, \\ &\tilde{A}_0 = A_0 + \partial_1\partial_\zeta X_+(0), \end{aligned} \quad (18)$$

satisfy $(\partial_1 - \zeta\partial_0)\tilde{Y}(\zeta) = \zeta\tilde{A}_0\tilde{Y}(\zeta)$, $\tilde{Y}(0) = 1$ and $\det \tilde{Y}(\zeta) = 1$. We call (18) the Riemann-Hilbert transformation. Following the general theory, one may convert (17) to an integral equation [10]. For the infinitesimal Riemann-Hilbert transformation induced from $u(\zeta) = \exp v(\zeta)$, we obtain

$$\begin{aligned} & Y(\zeta) \longrightarrow Y(\zeta) - \Xi(\zeta)Y(\zeta) \qquad \text{for } \zeta\ C_+, \\ & \Xi(\zeta) = \frac{1}{2\pi i}\oint_C \frac{d\zeta'}{\zeta'}\, Y(\zeta')v(\zeta')Y(\zeta')^{-1}. \end{aligned} \tag{19}$$

Then we have the following propositions.

<u>Proposition A</u> $\Xi(\zeta)$ satisfies

$$\begin{aligned} & (\partial_1 - \zeta\partial_0)\Xi(\zeta) = \zeta\{\partial_0 + [A_0,\ \chi + \Xi(\zeta)]\}, \\ & \chi = \frac{1}{2\pi i}\oint_C \frac{d\zeta'}{\zeta'}\, Y(\zeta')v(\zeta')Y(\zeta')^{-1}. \end{aligned} \tag{20}$$

<u>Proposition B</u> The infinitesimal Riemann-Hilbert transformation (19) induces a transformation on the potential A_0;

$$A_0 \longrightarrow A_0 - \{\partial_0\chi + [A_0,\ \chi]\}. \tag{21}$$

Let us define $\chi^{(n)}$ by (20) with $v(\zeta') = v\zeta'^{-n}$. Combining (16) and (20), we conclude that the infinitesimal Riemann-Hilbert transformations (19) induced from $v\zeta'^{-1}$ are identified with the infinitesimal transformations (13) up to the integration constants.

It is noted that the Riemann-Hilbert transformation (18) exponentiates the transformations proposed by Dolan et al.

2.3 Kac-Moody Lie Algebras

Refs. [8,9] also have computed the commutation relations $[M^{(n)}(v), M^{(m)}(v')] = M^{(n+m)}([v,v'])$ of the hidden symmetry algebras, where $M^{(n)}(v)$ are infinitesimal generators. Dolan [11] has pointed out that these algebras are isomorphic to the subalgebras $\mathfrak{g}\otimes\mathbb{C}[\zeta]$ of Kac-Moody algebras $\mathfrak{g}\otimes\mathbb{C}[\zeta,\zeta^{-1}]$.

On the other hand, the authors [10] have proved that the algebras of the infinitesimal Riemann-Hilbert transformations are isomorphic to Kac-Moody algebras. For the case of $\underline{G} = SL(n, \mathbb{C})$,

$$[v\zeta^{-n}, v'\zeta^{-m}] = [v,v']\zeta^{-(n+m)} \tag{22}$$

have been derived in [10]. Thus the Proposition A leads us to the hidden symmetry algebra $\mathfrak{sl}(n,\mathbb{C})\otimes\mathbb{C}[\zeta,\zeta^{-1}]$. The negative part $\mathfrak{sl}(n,\mathbb{C})\otimes\zeta^{-1}\mathbb{C}[\zeta^{-1}]$ does not appear in the framework of Refs. [8,9,11].

Finally we give some comments on SU(n) and SO(n) chiral fields. For SU(n) chiral fields, one has to impose the additional constraints on $Y(\zeta)$ and $u(\zeta)$ such that $Y(\zeta)^{\dagger}Y(\zeta) = 1$, $u(\zeta)^{\dagger}u(\zeta) = 1$. Here $\dagger$ stands for the hermitean conjugate of $Y(\zeta^*)$. The resultant Kac-Moody algebra is $\mathfrak{su}(n)\otimes\mathbb{R}[\zeta,\zeta^{-1}]$. For SO(n) case, constraints are $Y(\zeta)^{t}Y(\zeta) = 1$, $u(\zeta)^{t}u(\zeta) = 1$. Then $\mathfrak{so}(n)\otimes\mathbb{R}[\zeta,\zeta^{-1}]$ is the hidden symmetry algebra of SO(n) chiral fields.

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