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Topics on finite groups of characteristic 2 type

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Finite simple groups of characteristic 2 type have been classified by Aschbacher, Gorenstein, Lyons, G. Mason and others. In many places of their work, Aschbacher's $C(G,T)$-theorem plays an important role. Here, $C(G,T)$ is defined for each group G of even order and its Sylow 2-subgroup T by

$$C(G,T) = \langle N_G(C) \mid 1 \neq C \text{ char } T \rangle,$$

and the $C(G,T)$-theorem is a key to the classification of simple groups G of characteristic 2 type satisfying the condition

$$G \neq C(G,T).$$

The group $G = PSL_n(2^m)$ is a typical example of a group of characteristic 2 type. Let us see when G satisfies the above condition. G is the central factor group of the matrix group $SL_n(2^m)$. But I will ignore the center and regard G as $SL_n(2^m)$. We may choose as T the group of upper triangular matrices with all diagonal entries equal to 1. The center $Z(T)$ is the group of the matrices

$$\begin{pmatrix} 1,0,\cdots,0,x \\ \quad 1,0,\cdots,0 \\ \quad\quad \ddots \quad \vdots \\ \quad\quad\quad \ddots \; \vdots \\ 0 \quad\quad\quad 1,0 \\ \quad\quad\quad\quad 1 \end{pmatrix}$$

The Thompson subgroup $J(T)$ generated by the elementary abelian subgroups of maximal order consists of the matrices

$$\begin{pmatrix} I_k & X \\ 0 & I_k \end{pmatrix} \text{ when } n = 2k,$$

$$\begin{pmatrix} I_k & x & Y \\ & 1 & z \\ 0 & & I_k \end{pmatrix} \text{ when } n = 2k+1.$$

$N_G(Z(T))$ and $N_G(J(T))$ are easily determined, and we can see that if $n \geq 4$ then each minimal parabolic subgroup of G containing T is contained either in $N_G(Z(T))$ or in $N_G(J(T))$. So if $n \geq 4$ then $G = \langle N_G(Z(T)), N_G(J(T))\rangle$ and in particular $G = C(G,T)$. If $n = 2$ then T is elementary abelian and so $C(G,T) = N_G(T) \neq G$. If $n = 3$ then $Z(T)$ is the only non-trivial elementary characteristic subgroup of T and so $C(G,T) = N_G(Z(T)) \neq G$. Therefore, $G \neq C(G,T)$ if and only if $n = 2$ or 3

Now, if $n = 2$ then two distinct Sylow 2-subgroups of G intersect in the identity element. If $n = 3$, the maximal 2-local subgroups of G containing T are

$$M = \left\{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}\right\} \text{ and } N = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\}$$

M has a normal subgroup

$$K = \left\{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

which is the extension of the 2-dimensional vector space over $GF(2^m)$ by $SL_2(2^m)$, and N has an analogous normal subgroup.

The notions of <u>2-isolated group</u> and <u>Aschbacher</u>

block are related to the phenomenon described above. Let G be an arbitrary finite group, p a prime dividing the order of G, and S, T be its Sylow p-subgroups. If there is a sequence $\{S_i\}$ of Sylow p-subgroups joining S and T such that $S_{i-1} \cap S_i \neq 1$, then we say that S and T are equivalent. If G contains inequivalent Sylow p-subgroups, we say that G is p-isolated. Next, if a subnormal subgroup K of a group M satisfies the following conditions, we call K an Aschbacher block of M.

$$K = O^2(K),$$
$$\bar{K} = K/O_2(K) = SL_2(2^m) \text{ or } A_{2m-1},\ m > 1,$$
$$U = [O_2(M), K] \leqq \Omega_1(Z(O_2(K))), \text{ and}$$
$$V = U/C_U(K) \text{ is a natural } GF(2)\bar{K}\text{-module, i.e.}$$

either $\bar{K} = SL_2(2^m)$ and V is the 2-dimensional vector space over $GF(2^m)$ considered a $GF(2)\bar{K}$-module, or $\bar{K} = A_{2m-1}$ and V is the nontrivial composition factor of the $(2m-1)$-dimensional permutation module over $GF(2)$. The $C(G,T)$-theorem may now be stated.

Theorem 1. If G is a group of characteristic 2 type satisfying $G \neq C(G,T)$, then either

(1) some maximal 2-local subgroup of G has an Aschbacher block, or

(2) G is 2-isolated.

As well known, (2) implies that G has a strongly embedded subgroup and so the structure of G in the case (2) is known by Bender's theorem. The structure of G in the case (1) is determined by Foote, Solomon, and S. K. Wong. Thus, the C(G,T)-theorem says that with small number of exceptions, the factorization

$$G = < N_G(C) \mid 1 \neq C \text{ char } T >$$

holds provided G is of characteristic 2 type.

Now, in his talk at Santa Cruz Summer Institute (1979), McBride announced the following theorem.

Theorem 2. Let G be a group of characteristic 2 type, T a Sylow 2-subgroup of G, and suppose the set

$$\mathcal{N}(T) = \{N_G(C) \mid 1 \neq C \text{ char } T\}$$

does not control the fusion in T. Then some maximal 2-local subgroup of G has an Aschbacher block.

Here, the statement "$\mathcal{N}(T)$ controls the fusion in T" means that if A and B are two subsets of T which are conjugate in G then there exist members N_i of $\mathcal{N}(T)$, $1 \leqq i \leqq n$, and elements x_i of N_i such that

$$A^{x_1 x_2 \cdots x_n} = B \quad \text{and} \quad A^{x_1 x_2 \cdots x_i} \leqq T.$$

It is apparent that there is some relation between the above two theorems. What will it really be? An answer to this question will be obtained by the following consideration.

Let G be a group, p a prime dividing the order of G, and $\mathcal{H}_0$ the set of all nonidentity p-subgroups H such that $N_G(H)/H$ is p-isolated. Let $\mathcal{N}_0$ be the set of the normalizers of the members of $\mathcal{H}_0$ and for S in $Syl_p(G)$ let

$$\mathcal{N}_0(S) = \{N \in \mathcal{N}_0 \mid S \cap N \in Syl_p(N)\}$$

The fusion theorem of Alperin and Goldschmidt may be stated as follows.

Theorem 3. $\mathcal{N}_0$ controls the p-fusion in G.

Namely, if A and B are subsets of a Sylow p-subgroup S of G and are conjugate in G, then there exist members N_i of $\mathcal{N}_0(S)$, $1 \leqq i \leqq n$, elements x_i of N_i, and an element y of $N_G(S)$ such that

$$A^{x_1 x_2 \cdots x_n y} = B \quad \text{and} \quad A^{x_1 x_2 \cdots x_i} \leqq S \cap N_i.$$

(In the above, $S \cap N_i$ may be replaced by $O_p(N_i)$)

There are two theorems which relate $\mathcal{N}_0$ with factorizations and Sylow intersections.

Theorem 4. Unless G is p-isolated,

$$G = \langle \mathcal{N}_0(S),\ N_G(S) \rangle.$$

Theorem 5. $\mathcal{N}_0$ controls Sylow p-intersections in G.

Namely, if two distinct Sylow p-subgroups S and T intersect nontrivially, then there exist members N_i of $\mathcal{N}_0$, $1 \leqq i \leqq n$, and Sylow p-subgroups S_j of G, $0 \leqq j \leqq n$, such that

$S_0 = S$ and $S_n = T$,
$N_i \in \mathcal{N}_0(S_{i-1}) \cap \mathcal{N}_0(S_i)$,
$S_{i-1} = S_i^{x_i}$ for some $x_i \in N_i$, and
$S \cap T \leqq S_i \cap N_i$.

(In the above, $S_i \cap N_i$ may be replaced by $O_p(N_i)$)

What should be noticed here is that both Theorems 3 and 4 are easy consequences of Theorem 5. It goes without saying that

Theorems 1 and 2 are analogues of Theorems 4 and 3, respectively. If this analogy is not accidental, there should be an unknown theorem from which Theorems 1 and 2 are easily derived. And the unknown theorem should be as follows.

Theorem 6. Let G be a group of characteristic 2 type and suppose the set

$$\mathcal{N} = \{N_G(C) \mid 1 \neq C \text{ char } T,\ T \in \mathrm{Syl}_2(G)\}$$

does not control Sylow 2-intersections in G. Then some maximal 2-local subgroup of G has an Aschbacher block.

The meaning of "control" here is exactly the same as in Theorem 5; we just replace p and $\mathcal{N}_0$ by 2 and $\mathcal{N}$, respectively.

It turns out that Theorem 6 is really true and may be sharpened considerably. Let us consider the following general situation. Let G be a group, p a prime dividing the order of G, and $\mathcal{F}$ a normal set of subgroups of G (i.e. $\mathcal{F}$ is closed under conjugation). For Sylow p-subgroups S of G, let

$$\mathcal{F}(S) = \{X \in \mathcal{F} \mid S \cap X \in \mathrm{Syl}_p(X)\}$$

If the conclusion of Theorem 5 is true with $\mathcal{N}_0$ replaced by $\mathcal{F}$, we say that $\mathcal{F}$ controls Sylow p-intersections in G. If the conclusion of Theorem 3 is true with $\mathcal{N}_0$ replaced by $\mathcal{F}$, we say that $\mathcal{F}$ controls the p-fusion in G. The following result correlates these two kinds of "control" and "factorization".

Proposition 1. If $\mathcal{F}$ controls Sylow p-intersections in G, then $\mathcal{F}$ controls the p-fusion in G and, unless G is p-isolated, $G = \langle \mathcal{F}(S), N_G(S) \rangle$.

Typical examples of $\mathcal{F}$ are found in the following situation.

Hypothesis 1. To each nonidentity p-subgroup P of G there is associated a set $f(P)$ of subgroups of P such that $N_G(P) \leqq N_G(F)$ for each F in $f(P)$ and $f(P)^g = f(P^g)$ for each g in G.

For each subgroup H of order divisible by p, let

$$\mathcal{F}_H = \{N_H(F) \mid F \in f(P),\ P \in \mathrm{Syl}_p(H)\}$$

$\mathcal{F}_H$ is a normal set of subgroups of H. Let $\mathcal{F} = \mathcal{F}_G$ and denote by $\mathcal{F}'$ the set of all maximal p-local subgroups M such that $\mathcal{F}_M$ does not control Sylow p-intersections in M. $\mathcal{F}'$ is also a normal set. Then we have the following result.

Proposition 2. Under Hypothesis 1, if $\{1\} \notin f(P)$ for each P, then $\mathcal{F} \cup \mathcal{F}'$ controls Sylow p-intersections in G.

Let us specialize to the following situation.

Hypothesis 2. Hypothesis 1 with $f(P) = \{F_1, F_2, F_3\}$, $F_i \neq 1$, and for $j = 2, 3$, $F_1 \leqq C_P(F_j)$ and $N_G(C_P(F_j)) \leqq N_G(F_1)$. Let

$$\mathcal{E} = \{N_G(F_1(S)),\ C_G(F_2(S) \cap F_3(S)) \mid S \in \mathrm{Syl}_p(G)\},$$

where $f(S) = \{F_i(S) \mid i = 1, 2, 3\}$.

The following result is a consequence of Proposition 2.

Proposition 3. Under Hypothesis 2, $\mathcal{G} = \mathcal{E} \cup \mathcal{F}'$ controls Sylow p-intersections in G.

By Proposition 1, $\mathcal{G}$ controls the p-fusion in G and, unless G is p-isolated, $G = \langle \mathcal{G}(S) \rangle$ for each Sylow p-subgroup S of G. Hence, if we can find a mapping f which satisfies Hypothesis 2 with $\mathcal{F}'$ small, we get good information about the Sylow p-intersection, p-fusion, and p-factorization. The following result guarantees the existence of such f for $p = 2$ and groups of characteristic 2 type.

Theorem 7. Each nonidentity 2-group P has nonidentity characteristic subgroups A_P and B_P satisfying the following conditions.

(1) A_P char $C_P(B_P)$,

(2) If a group M satisfies $C_M(O_2(M)) \leqq O_2(M)$ and if the set

$$\{N_M(A_S),\ N_M(B_S),\ N_M(\Omega_1(Z(S))) \mid S \in Syl_2(M)\}$$

does not control Sylow 2-intersections in M, then M has an Aschbacher block.

The mapping

$$f(P) = \{A_P,\ B_P,\ \Omega_1(Z(P))\}$$

satisfies Hypothesis 2 and so if

$$\mathcal{E} = \{N_G(A_T),\ C_G(B_T \cap \Omega_1(Z(T)) \mid T \in Syl_2(G)\},$$

then $\mathcal{G} = \mathcal{E} \cup \mathcal{F}'$ controls Sylow 2-intersections in G by Proposition 3. $\mathcal{F}'$ is the set of all maximal 2-local subgroups M of G such that $\mathcal{F}_M$ does not control Sylow 2-intersections in M. Hence if G is of characteristic 2 type, each member of $\mathcal{F}'$ has an Aschbacher block by (2) of Theorem 7. This proves the following.

Corollary 1. If G is a group of characteristic 2 type, then Sylow 2-intersections and the 2-fusion in G are controlled by $\mathcal{E}$ and maximal 2-local subgroups having Aschbacher blocks. In particular, Theorems 2 and 6 hold.

Corollary 2. Let G be a group of characteristic 2 type, T a Sylow 2-subgroup of G, and suppose G is not 2-isolated. Then G is generated by $N_G(A_T)$, $C_G(B_T \cap \Omega_1(Z(T)))$, and maximal 2-local subgroups M having Aschbacher blocks with $T \cap M \in Syl_2(M)$. In particular, Theorem 1 holds.

This corollary also contains as a special case a variant of Theorem 1 obtained by Aschbacher. I can not at present explicitly give A_P and B_P in general. However, if the nilpotence class of

$$K(P) = C_P(\Omega_1(Z(J(P))))$$

is at most 2, then we may choose as A_P and B_P an arbitrary nonidentity characteristic subgroup of $K(P)$ and an arbitrary nonidentity characteristic subgroup of $K(P)$ contained in $\Omega_1(Z(K(P))) = \Omega_1(Z(J(P)))$. Therefore, the following holds.

Corollary 3. Let G be a group of characteristic 2 type and suppose, for $T \in Syl_2(G)$, $K(T)$ has class at most 2. Let

$$\mathcal{E} = \{N_G(K(T)),\ C_G(\Omega_1(Z(T)))\ |\ T \in Syl_2(G)\}$$

Then Sylow 2-intersections and the 2-fusion in G are controlled by $\mathcal{E}$ and maximal 2-local subgroups having Aschbacher blocks.

Corollary 4. Under the hypothesis of Corollary 3, if $T \in Syl_2(G)$, then G is generated by $N_G(K(T))$, $C_G(\Omega_1(Z(T)))$, and maximal 2-local subgroups M having Aschbacher blocks with $T \cap M \in Syl_2(M)$.

$G = PSL_n(2^m)$ satisfies the hypothesis of Corollary 3 and if $n \geqq 4$ then there are no Aschbacher blocks in maximal 2-local subgroups. Moreover, $K(T) = J(T)$ and $\Omega_1(Z(T)) = Z(T)$ for T in $Syl_2(G)$. Hence if $n \geqq 4$ then Sylow 2-intersections and the 2-fusion in G are controlled by $N_G(J(T))$ and $N_G(Z(T))$. Also, $G = \langle N_G(J(T)), N_G(Z(T)) \rangle$ if $n \geqq 4$. These facts are also verified by using the fact that each minimal parabolic subgroup containing T is contained in $N_G(J(T))$ or $N_G(Z(T))$. I think this a good supporting evidence for Theorem 7.

The theorems 2, 6, 7, and the corollaries are, in fact, proved under the hypothesis that each proper simple section of the group under consideration is of known type; that is, isomorphic to one of the alternating groups, the Lie type groups, and the sporadic groups. I have ignored the hypothesis in this exposition because finite simple groups have been classified and, in the classification program, the above theorems are designed to be applied to minimal unknown simple groups.

I will conclude this expository article by a rough sketch of the proof of Theorem 7. For each nonidentity 2-group S, let $\mathbb{G}(S)$ denote the collection of all finite groups G satisfying the following conditions.

(1) $S \in Syl_2(G)$,

(2) $C_G(O_2(G)) \leqq O_2(G)$,

(3) $\overline{G} = G/C_G(\Omega_1(Z(O_2(G)))) = SL_2(2^m)$,

(4) when $V = \Omega_1(Z(O_2(G)))$ is regarded as a $GF(2)\overline{G}$-module, $[V,\overline{G}]/C_{[V,\overline{G}]}(\overline{G})$ is a natural module,

(5) $O_2(G) \in Syl_2(C_G(V))$,

(6) $[O_2(G), O^2(G)] \nleqq V$,

(7) S is contained in a unique maximal subgroup of G,

(8) $G = \langle K(S)^G \rangle$.

The conditions (4) and (5) are derived from other conditions; I have listed them for informational purposes. Also, if $\mathbb{G}(S)$ is nonempty then $S = K(S)$ and the nilpotence class of S is at least 3.

A <u>characteristic pair</u> for the 2-group S is a pair S_1, S_2 of characteristic subgroups of S such that whenever $G \in \mathbb{G}(S)$, either S_1 or S_2 is normal in G. The characteristic pair is said to be nontrivial if $S_1 \neq 1 \neq S_2$. A work of Campbell shows that for each nonidentity 2-group S there exists a nontrivial characteristic pair satisfying $S_1 \leqq \Omega_1(Z(S))$. I say such a pair is of Glauberman-Niles type.

Now, for each nonidentity 2-group S satisfying $S = K(S)$, fix a characteristic pair S_1, S_2 of Glauberman-Niles type. For an arbitrary nonidentity 2-group P, define

$$A_P = (K(P))_1 \quad \text{and} \quad B_P = (K(P))_2$$

Then A_P and B_P satisfy the condition (1) of Theorem 7, and our aim is to prove (2) of Theorem 7 with this choice of A_P and B_P.

Let M be a minimal counterexample to (2) of Theorem 7. Let us first consider the special case where $M/O_2(M)$ is a non-abelian simple group Let $Q = O_2(M)$ and $V = \Omega_1(Z(Q))$. Then

$$C_M(V) = Q$$

If $C_M(V) \neq Q$, then $C_M(V) = M$ by the simplicity of M/Q, and so $M = N_M(\Omega_1(Z(S)))$ for $S \in Syl_2(M)$ as $\Omega_1(Z(S)) \leqq C_M(Q) \leqq Q$. But this contradicts our assumption that the set

$$\mathcal{F} = \{N_M(A_S),\ N_M(B_S),\ N_M(\Omega_1(Z(S)))\ |\ S \in Syl_2(M)\}$$

does not control Sylow 2-intersections in M. Next, we have

$$J(S) \not\leq Q$$

If $J(S) \leq Q$, then $V \leq \Omega_1(Z(J(S)))$, $K(S) \leq C_M(V) = Q$, and so $K(S) = K(Q)$. But then $M = N_M(A_S) = N_M(B_S)$, a contradiction. Let $\overline{M} = M/Q$. Then $\overline{M}$ is faithfully represented on the GF(2)-space V. As $J(S) \not\leq Q$, there is an elementary abelian 2-subgroup A of maximal order such that $\overline{A} \neq 1$. The maximality of $|A|$ yields that $|\overline{A}| \geq |V : C_V(\overline{A})|$. A work of Aschbacher on GF(2)-representations with this property shows that $\overline{M}$ is either an alternating group or a group of Lie type and even characteristic but not $Sz(2^m)$ or $PSU_3(2^m)$. Suppose for instance that $\overline{M}$ is a Lie type group of rank at least 2. The BN-pair structure of $\overline{M}$ shows that Sylow 2-intersections in $\overline{M}$ are controlled by minimal parabolic subgroups. Hence there must exist a minimal parabolic subgroup $\overline{X} = X/Q$ such that Sylow 2-intersections in X are not controlled by the set

$$\{N_X(A_S),\ N_X(B_S),\ N_X(\Omega_1(Z(S)))\ |\ S \in Syl_2(X)\}$$

Since M is a minimal counterexample, it follows that X has an Aschbacher block K. But since $C_M(Q) \leq Q$ and $C_{\overline{X}}(O_2(\overline{X})) \leq O_2(\overline{X})$, $O_2(X)$ has at least two nontrivial K-chief factors, contrary to the definition of the Aschbacher block. Therefore, if $\overline{M}$ is of Lie type then $\overline{M} = SL_2(2^m)$. Moreover, the presence of $\overline{A}$ shows that $[V,\overline{M}]/C_{[V,\overline{M}]}(\overline{M})$ is a natural module. As M cannot be an Aschbacher block of M, we must have

$$[Q,\ O^2(M)] \not\leq V.$$

In this way, the structure of the minimal counterexample M is highly restricted, and we can eventually show that $M \in \mathbb{G}(S)$ and $S = K(S)$. But then either A_S or B_S is normal in M, which contradicts our assumption that F does not control Sylow 2-intersections in M. This completes the proof when $\bar{M}$ is of Lie type and even characteristic. I will not discuss the case where $\bar{M}$ is an alternating group. Difficulties arise when $M/O_2(M)$ is not simple. However, the basic idea of the proof is simple as illustrated above.

In the general case, we have $O_2(M/C_M(V)) = 1$ and

$$Q \in Syl_2(C_M(V))$$

and so the previous argument shows

$$J(S) \not\leqq C_M(V).$$

Then $\bar{M} = M/C_M(V)$ is faithfully represented on the GF(2)-space V, and $\bar{M}$ contains an elementary abelian 2-subgroup $\bar{A} \neq 1$ such that $|\bar{A}| \geqq |V : C_V(\bar{A})|$. The theory of GF(2)-representation with this property and the minimality of M show that $\bar{M}$ has a normal subgroup $\bar{N} = N/C_M(V)$ such that either $\bar{N}$ is a direct product of $SL_2(2)$'s or $\bar{N}$ is a central product of the conjugates of a subgroup $\bar{L}$ which is a perfect central extension of an alternating group or a group of Lie type and even characteristic but not $Sz(2^m)$ or $PSU_3(2^m)$ and such that $\bar{M}/\bar{N}$ is a 2-group. Moreover, if $\bar{L}/Z(\bar{L}) = PSL_3(2^m)$ or $Sp_4(2^m)$ then each element of $N_{\bar{M}}(\bar{L})$ induces an inner or graph or field automorphism on $\bar{L}$. Now, the previous argument and the minimality of M show that Sylow 2-intersections in $\bar{M}$ are not controlled by proper subgroups $\bar{X}$ of odd index satisfying $C_{\bar{X}}(O_2(\bar{X})) \leqq O_2(\bar{X})Z(\bar{N})$. Thus, we are led to the following situation.

Hypothesis 3. G is a finite group, N is a normal subgroup, G/N is a 2-group, N is a central product of the groups $L = L_1, L_2, \cdots, L_k$ which are all conjugate in G, and L is a perfect central extension of either an alternating group or a group of Lie type and even characteristic.

We wish to know when Sylow 2-intersections in G are controlled by proper subgroups X of odd index such that $C_X(O_2(X)) \leqq O_2(X)Z(N)$. An answer to this question may be obtained by the following consideration. Let G be an arbitrary finite group, and $\mathcal{H}_0$ the set defined before for a prime p. For each H in $\mathcal{H}_0$, $N_G(H)/H$ has a unique minimal subnormal subgroup $N_G^*(H)/H$ of order divisible by p. Then we have

Proposition 4. A normal set $\mathcal{F}$ of subgroups of G controls Sylow p-intersections in G provided that for each H in $\mathcal{H}_0$, $N_G^*(H)$ is contained in some member of $\mathcal{F}$.

Therefore, given a group G satisfying Hypothesis 3, we need to know when each $N_G^*(H)$ for $p = 2$ is contained in some subgroup X of odd index satisfying $C_X(O_2(X)) \leqq O_2(X)Z(N)$. Let us consider the case where $L/Z(L)$ is a group of Lie type and even characteristic. Using the knowledge about the 2-local subgroups of L, we can show that if some $N_G^*(H)$ is not contained in any such subgroup then either $L/Z(L)$ is of rank 1 or $L/Z(L)$ is $PSL_3(2^m)$ or $Sp_4(2^m)$ and some element of $N_G(L)$ induces a graph or graph-field automorphism on L.

Returning to the previous notation, we can see now that if $\bar{L}/Z(\bar{L})$ is a group of Lie type and even characteristic then $\bar{L} = SL_2(2^m)$. The next step in the proof of (2) is to show

$$M = \langle K(S)^M \rangle$$

using the minimality of M. This is technically the most difficult part of the proof, but I will not discuss it. Let us assume $\bar{L} = SL_2(2^m)$. Then the theory of GF(2)-representations show that $\overline{K(S)}$ normalizes $\bar{L}$ and if $\bar{N} = \bar{L}$ then $\overline{K(S)} \leqq \bar{L}$. Therefore,

$$\bar{M} = \bar{L}.$$

A short argument now shows that $M \in \mathbb{G}(S)$ and $S = K(S)$. This completes the proof when $\bar{L}/Z(\bar{L})$ is a group of Lie type and even characteristic. The argument for the alternating groups is essentially the same but is longer because the 2-local structure of the alternating groups is more complicated than the groups of Lie type and even characteristic.

Finally, I emphasize that the work of M. Aschbacher and P. McBride have had a great influence on my work described above. Particularly, I have learned much from a series of papers by M. Aschbacher listed below.

References

(1) M. Aschbacher, A factorization theorem for 2-constrained groups, J. London Math. Soc. (1981),

(2) _____, On the failure of the Thompson factorization in 2-constrained groups, J. London Math. Soc. (1981),

(3) _____, GF(2)-representations of finite groups, Amer. J. Math. 104 (1982), 683 - 771.

(4) _____, D. Gorenstein, and R. Lyons, The embedding of 2-locals in finite groups of characteristic 2-type I, II, Ann. of Math. 114 (1981), 335 - 456.

(5) H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festlasst, J. Algebra 17 (1971), 525 - 554.

(6) N. R. Campbell, Pushing up in finite groups, Ph. D. Thesis, California Institute of Technology, 1979.

(7) R. Foote, Component type theorems for finite groups in characteristic 2, Ill. J. Math. 26 (1982), 62 - 111.

(8) G.Glauberman and R. Niles, A pair of characteristic subgroups for pushing-up in finite groups, to appear.

(9) D. Goldschmidt, A conjugation family for finite groups, J. Algebra 16 (1970), 138 - 142.

(10) K. Gomi, Sylow 2-intersections, 2-fusion, and 2-factorizations in finite groups of characteristic 2 type, preprint.

(11) R. Solomon, On certain 2-local blocks, to appear.

(12) _____ and S. K. Wong, On $L_2(2^n)$-blocks, to appear.