On topological Blaschke conjecture I

Cohomological complex projective spaces

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By a Blaschke manifold, we mean a Riemannian manifold \((M,g)\) such that, for any point \(m \in M\), the tangential cut locus \(C_m\) of \(m\) in \(T_mM\) is isometric to the sphere of constant radius. There are some equivalent definitions (see Besse[2, 5.43]). The Blaschke conjecture is that any Blaschke manifold is isometric to a compact rank one symmetric space. If the integral cohomology ring of \(M\) is equal to the sphere \(S^k\), or the real projective space \(RP^k\), this conjecture is proved by Berger with other mathematicians [2, Appendix D]). We consider the case where the cohomology ring of \(M\) is equal to that of the complex projective space \(CP^k\).

We obtain the following theorem.

Theorem. Let \((M,g)\) be a 2k-dimensional Blaschke manifold such that the integral cohomology ring is equal to that of \(CP^k\). Then \(M\) is PL-homeomorphic to \(CP^k\) for any \(k\).

Blaschke manifolds with other cohomology rings will be treated in subsequent papers.
If \((M,g)\) is a Blaschke manifold and \(m \in M\), Allamigeon [1] has shown that the cut locus \(C(m)\) of \(m\) in \(M\) is the base manifold of a fibration of the tangential cut locus \(C_m\) by great spheres. We study the base manifold of such fibration by great circles. We apply the Browder-Novikov-Sullivan's theory in the classification of homotopy equivalent manifolds (see Wall[4]). Calculation of normal invariants gives our theorem. In Appendix, we give examples of non-trivial fibrations of \(S^3\) by great circles. The author thanks to M.Mizutani and K.Masuda for the discussion of results in Appendix.

Detailed proof will appear elsewhere.
§1. Projectable bundles

In the paper [3], we have obtained a method of calculation of the tangent bundle of the base space of an $S^1$-principal bundle. We will briefly recall that.

Let $X$ be a smooth manifold and let $\pi : L \rightarrow X$ be the projection of an $S^1$-principal bundle.

**Definition.** A vector bundle $p : E \rightarrow L$ over $L$ is projectable onto $X$, if there exists a vector bundle $\hat{p} : \hat{E} \rightarrow X$ over $X$ such that $\pi^*\hat{E} = E$. The map $\pi$ induces the bundle map $\pi_! : E \rightarrow \hat{E}$, which we call the projection. The bundle $\hat{E}$ is called the projected bundle.

Let $x$ be a point in $X$. For any $a, b \in \pi^{-1}(x) = S^1$, we have a linear isomorphism

$$\phi_{ab} : p^{-1}(a) \rightarrow p^{-1}(b)$$

of vector spaces defined by $\phi_{ab}(u) = v$, where $\pi_!(u) = \pi_!(v)$. Then we have, for $a, b, c \in \pi^{-1}(x)$,

$$(1) \quad \phi_{bc} \phi_{ab} = \phi_{ac}$$

Let $\pi^*L = \{(a, b) \in L \times L, \pi(a) = \pi(b)\}$ be the induced $S^1$-bundle over $L$ from $L$. We have two projections $\pi_1$,

$$\pi_2 : \pi^*L \rightarrow L$$

defined by $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

Let $\pi_i^*E$ ($i = 1, 2$) be the induced vector bundle. The map $\phi : \pi^*L \rightarrow \text{Iso}(\pi_1^*E, \pi_2^*E)$ defined by $\phi(a, b) = \phi_{ab}$ is a continuous cross section of the bundle $\text{Iso}(\pi_1^*E, \pi_2^*E)$ over $\pi^*L$.
We call $\phi$ the projecting isomorphism associated with the projectable bundle $E$.

Proposition 1. Suppose given a vector bundle $E$ over $L$ and a cross section $\phi$ of the bundle $\text{Iso}(\pi_1^*E, \pi_2^*E)$ satisfying (1). Then we have a vector bundle $\hat{E}$ over $X$ such that $\pi^*\hat{E} = E$ and the projecting isomorphism is equal to $\phi$.

Now let $TL$ and $TX$ be the tangent bundles of $L$ and $X$ respectively. Let $\rho : S^1 \times L \rightarrow L$ be the free $S^1$-action. For each $t \in S^1$, the diffeomorphism $\rho(t) = \rho(t, \cdot)$ induces a bundle isomorphism $\rho(t)_* : TL \rightarrow TL$.

Proposition 2. The collection $\bigcup_{t \in S^1} \rho(t)_*$ induces a projecting isomorphism on the bundle $TL$ such that the projected bundle $TL$ is isomorphic to $TX \oplus 1$.

Proof. Choose a bundle metric on $TL$. Let $TL_1$ be the subbundle of $TL$ consisting of tangent vectors normal to the $S^1$-action. Then $TL_1$ is projected to $TX$. The line bundle tangent to the $S^1$-action is projected to the trivial line bundle on $X$. 

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§2. Pontrjagin classes

Let \( S^{2k-1} \) be the unit sphere in \( \mathbb{R}^{2k} \) and let \( \pi: S^{2k-1} \to B \) be a fibration of \( S^{2k-1} \) by great circles. Thus, for each \( b \in B \), \( \pi^{-1}(b) \) is the intersection of \( S^{2k-1} \) with a 2-plane in \( \mathbb{R}^{2k} \). We write the 2-plane by \( P(b) \). Let \( \rho: S^1 \times S^{2k-1} \to S^{2k-1} \) denote the free \( S^1 \)-action.

Let \( V(2k,2) \) and \( G(2k,2) \), respectively, be the Stiefel and the Grassmann manifold consisting of orthogonal 2 frames or oriented 2-planes in \( \mathbb{R}^{2k} \). Then the natural mapping \( \lambda: V(2k,2) \to G(2k,2) \) defines a principal \( S^1 \)-bundle.

The mapping \( \theta: B \to G(2k,2) \) defined by \( \theta(b) = P(b) \) is a smooth embedding. Let \( \theta^*(\lambda) \) denote the induced bundle of \( \lambda \) by \( \theta \). Since \( \pi \) is also the induced bundle of \( \lambda \) by \( \theta \), there exists a natural bundle isomorphism between \( \pi \) and \( \theta^*(\lambda) \) inducing the identity on \( B \). Thus we obtain:

Lemma 3. We may suppose that the free \( S^1 \)-action \( \rho \) on \( S^{2k-1} \) is equal to the restriction on \( \pi^{-1}(b) \) of the linear action on \( P(b) \) for every \( b \in B \).

In the following, we always assume that \( \rho \) is the linear action on each fibre. For each \( x \in S^{2k-1} \), let \( Kx \) denote the point \( \rho(1/4)x \) in \( S^{2k-1} \), where we identify
$S^1$ with $[0,1]/[0] \sim [1]$. Define a mapping $\psi : S^{2k-1} \rightarrow V(2k,2)$ by $\psi(x) = (x, Kx)$. This is a smooth embedding and is a bundle map inducing $\theta$ on the base manifolds. For an orthogonal 2-frame $w = (x,y)$, let $\hat{\psi}(w)$ denote the vector $(x/\sqrt{2}, y/\sqrt{2})$ in $\mathbb{R}^{2k} \oplus \mathbb{R}^{2k}$. Then the map $\hat{\psi} : V(2k,2) \rightarrow \mathbb{R}^{4k}$ is a smooth embedding of $V(2k,2)$ in $S^{4k-1} \subset \mathbb{R}^{4k}$. We identify $\mathbb{R}^{2k} \oplus \mathbb{R}^{2k}$ with $\mathbb{C}^{2k}$ such that the first summand $\mathbb{R}^{2k}$ is the real part and the second pure imaginary. On $\mathbb{C}^{2k} - 0$, we have the free action $\rho_0$ of $S^1$ as the multiplication by the complex number of norm one. Then $\hat{\psi}$ is $S^1$-equivalent and we write by $\psi$ the induced map $\psi : G(2k,2) \rightarrow \mathbb{C}P^{2k-1}$.

Let $\tilde{f} : S^{2k-1} \rightarrow S^{4k-1}$ be the composition $\tilde{f} = \psi \theta$ and $f = \psi \theta : B \rightarrow \mathbb{C}^{2k-1}$. The map $\tilde{f}$ is given by $\tilde{f}(x) = (x/\sqrt{2}, Kx/\sqrt{2})$ for $x \in S^{2k-1}$.

We define a map $\tilde{F} : \mathbb{R}^{2k} - 0 \rightarrow \mathbb{C}^{2k} - 0$ by $\tilde{F}(tx) = t\tilde{f}(x)$ for $t > 0$ and $x \in S^{2k-1}$. The map $\tilde{F}$ is a smooth embedding. Let $E$ denote the restriction of the tangent bundle $T(\mathbb{R}^{2k} - 0)$ of $\mathbb{R}^{2k} - 0$ on $S^{2k-1}$, and we write $p$ for the projection $E \rightarrow S^{2k-1}$. Then $\tilde{F}$ induces an injective bundle map $\tilde{F}_* : E \rightarrow \tilde{F}_*(E) \subset T(\mathbb{C}^{2k} - 0)|_{\tilde{F}(S^{2k-1})}$.

Now define a map $\tilde{G} : \mathbb{R}^{2k} - 0 \rightarrow \mathbb{C}^{2k} - 0$ by $\tilde{G}(tx) = (tx/\sqrt{2}, -tK/\sqrt{2})$ for $t > 0$ and $x \in S^{2k-1}$.
Then $\tilde{\mathcal{G}}$ is also an embedding and $\tilde{\mathcal{G}}$ induces an injective bundle map

$$\tilde{\mathcal{G}}_* : E \to \tilde{\mathcal{G}}_*(E) \subset T(\mathbb{C}^{2k-0} - 0)|_{\tilde{\mathcal{G}}(S^{2k-1})}.$$ 

If $\tilde{\rho}_0$ denote the conjugate action of $S^1$ on $\mathbb{C}^{2k-0}$. Then $G$ is $S^1$-equivariant concerning to this conjugate action.

For any $y \in \mathbb{C}^{2k}$, we naturally identify the tangent space $T_y \mathbb{C}^{2k}$ with $\mathbb{C}^{2k}$ itself. For $x \in S^{2k-1}$, let $E_x$ denote the fiber $p^{-1}(x)$. Then $\tilde{F}_*(E_x)$ and $\tilde{G}_*(E_x)$ are subvector spaces of $\mathbb{C}^{2k}$.

Since $K : S^{2k-1} \to S^{2k-1}$ is a diffeomorphism, we obtain:

**Lemma 4.** The vector spaces $\tilde{F}_*(E_x)$ and $\tilde{G}_*(E_x)$ are transversal. Thus they span $\mathbb{C}^{2k}$.

Let $T$ denote the restriction of the tangent bundle $T(\mathbb{C}^{2k})$ on $\tilde{F}(S^{2k-1})$. Then we have the direct sum decomposition by trivial vector bundles

$$T = \tilde{F}_*(E) \oplus \tilde{G}_*(E).$$

Notice that $\tilde{G}_*(E)$ on $\tilde{G}(S^{2k-1})$ is identified with the subbundle in $T$ over $\tilde{F}(S^{2k-1})$ by an orientation reversing diffeomorphism of $S^{2k-1}$.

For any $t \in S^1$, we have the induced bundle isomorphisms $\rho_*(t) : E \to E$ and $\rho_0(t) : T \to T$. 

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Lemma 5. The isomorphism $\rho_0^*(t)$ is equal to the direct sum

$$\rho_\ast(t) + \rho_\ast(t).$$

By Proposition 1, we obtain that the projected bundle $\hat{T}$, defined by the projecting isomorphism $\rho_\ast(t)$, is isomorphic to the Whitney sum;

$$\hat{T} \cong \hat{E} \oplus \hat{E}.$$

On the other hand, by Proposition 2, we obtain the following.

Lemma 6. The bundle $\hat{T}$ has the complex structure. As a complex vector bundle, $\hat{T}$ is isomorphic to the Whitney sum

$$T(\mathbb{C}P^{2k-1})|_{f(B)} \oplus 1.$$

Lemma 7. As a real vector bundle, $\hat{E}$ is isomorphic to the bundle $T(B) \oplus 2$.

Consequently, we obtain that

$$T(B) \oplus T(B) \oplus 4 \cong (T(\mathbb{C}P^{2k-1})|_{f(B)} \oplus 1)\mathbb{R}.$$

Since the cohomology groups $H^*(B;\mathbb{Z})$ has no torsion element, by the product formula of Pontrjagin classes, we obtain the following.

Proposition 8. The Pontrjagin classes of the smooth manifold $B$ is equal to that of $\mathbb{C}P^{k-1}$, for any $k$. 

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§3. $Z_2$-invariants and proof of Theorem

Let $\mathcal{J}(\mathbb{C}P^{k-1})$ denote the set of PL-homeomorphism classes of closed PL-manifolds homotopy equivalent to $\mathbb{C}P^{k-1}$. The following results are due to Sullivan (cf. [4, §14 C]).

Suppose that $k > 3$.

Proposition 9. For any $N \in \mathcal{J}(\mathbb{C}P^{k-1})$, there are invariants $s_{4i+2}(N) \in Z_2$ and $s_{4j}(N) \in Z$, for all integers $i,j$ satisfying $6 \leq 4i+2 < 2(k-1)$, $4 \leq 4j < 2(k-1)$. The invariants define a bijection of $\mathcal{J}(\mathbb{C}P^{k-1})$ with

$$( \bigoplus_{i} Z_2 ) \bigoplus ( \bigoplus_{j} Z )$$

The invariants $s_{4j}$ satisfy the following relations.

Proposition 10. If all the Pontrjagin classes of $N$ in $\mathcal{J}(\mathbb{C}P^{k-1})$ coincide with that of $\mathbb{C}P^{k-1}$, then $s_{4j}(N) = 0$ for all $j$.

Concerning $Z_2$-invariants $s_{4i+2}$, the following holds. Let $\mathcal{J}(\mathbb{R}P^{2k-1})$ denote the set of PL-homeomorphism classes of closed PL-manifolds homotopy equivalent to $\mathbb{R}P^{2k-1}$. This set is known to be equal to the isomorphism classes of homotopy triangulations of $\mathbb{R}P^{2k-1}$. Any $N \in \mathcal{J}(\mathbb{C}P^{k-1})$ is the base manifold of a PL free $S^1$-action on $S^{2k-1}$. By restricting the action to $Z_2 = S^0 \subset S^1$, we obtain a manifold homotopy equivalent to $\mathbb{R}P^{2k-1}$. 

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This defines a map
\[ \pi^b : \mathcal{A}(\mathbb{C}P^{k-1}) \longrightarrow \mathcal{A}(\mathbb{R}P^{2k-1}) . \]

The following holds ([4, §14D.3]).

Proposition 11. Let \( N \) be an element in \( \mathcal{A}(\mathbb{C}P^{k-1}) \) such that \( \pi^b(N) \) is PL-homeomorphic to \( \mathbb{R}P^{2k-1} \). Then
\[ s_{4i+2}(N) = 0 , \]
for all \( i \).

Now let \( B \in \mathcal{A}(\mathbb{C}P^{k-1}) \) be the base manifold of the fibration of \( S^{2k-1} \) by great circles. Then, obviously, the image \( \pi^b(B) \in \mathcal{A}(\mathbb{R}P^{2k-1}) \) is PL-homeomorphic to \( \mathbb{R}P^{2k-1} \).

Combining the result of §2 with Propositions, we obtain:

Proposition 12. The base manifold \( B \) of a fibration of \( S^{2k-1} \) by great circles is PL-homeomorphic to \( \mathbb{C}P^{k-1} \) if \( k \neq 3 \).

Now let us prove Theorem. Since the integral cohomology ring of \( M \) is equal to that of \( \mathbb{C}P^k \), \( M \) is simply connected ([2, 7.23]). Thus \( M \) is homotopy equivalent to \( \mathbb{C}P^k \). By Allamigeon's theorem, we know that \( M \) is PL-homeomorphic to the union of the disc \( D^{2k} \) with the \( D^2 \)-bundle associated with the fibration of \( S^{2k-1} \) by great circles. We write \( B \) for the base manifold of the fibration. If \( k = 3 \), by Proposition 9, \( M \) is
PL-homeomorphic to $\mathbb{C}P^3$ if and only if $s_4(M) = 0$.

The invariant $s_4(M)$ is calculated from the first Pontrjagin class $p_1(B)$ of $B$. By Proposition 8 of §2, $p_1(B)$ is equal to $p_1(\mathbb{C}P^2)$. Thus we have $s_4(M) = 0$ and $M$ is PL-homeomorphic to $\mathbb{C}P^3$. Now suppose that $k \neq 3$. According to Proposition 12, $B$ is PL-homeomorphic to $\mathbb{C}P^{k-1}$. The Euler class of the $S^1$-bundle is equal to a generator of $H^2(\mathbb{C}P^{k-1}; \mathbb{Z}) = \mathbb{Z}$. Thus the total space of the $D^2$-bundle is PL-homeomorphic to the tubular neighborhood of $\mathbb{C}P^{k-1}$ in $\mathbb{C}P^k$. Any orientation preserving PL-homeomorphism of $S^{2k-1}$ is isotopic to the identity. The attached manifold $M$ is PL-homeomorphic to $\mathbb{C}P^k$, which completes the proof of Theorem.
§4. Appendix

If \( \pi : S^{2k-1} \rightarrow B \) is a fibration by great circles, we get the embedding \( \theta : B \rightarrow G(2k,2) \). Since the planes \( \theta(b) \) for all \( b \in B \) give a foliation of \( S^{2k-1} \), we have the following property.

(*) For two different points \( b \) and \( b' \) in \( B \), the planes \( \theta(b) \) and \( \theta(b') \) are transverse.

The converse holds.

Lemma 13. Let \( \pi : S^{2k-1} \rightarrow B \) be a principal \( S^1 \)-bundle induced from the \( S^1 \)-bundle \( \lambda : W(2k,2) \rightarrow G(2k,2) \) by a smooth embedding \( \theta : B \rightarrow G(2k,2) \). Suppose that, for any different points \( b \) and \( b' \) in \( B \), the planes \( \theta(b) \) and \( \theta(b') \) are transversal. Then the bundle \( \pi \) is a fibration of \( S^{2k-1} \) by great circles.

Proof. Consider the union \( \bigcup_{b} (\theta(b) \cap S^{2k-1}) \).

Then it covers \( S^{2k-1} \) and give a fibration by great circles.

Now we consider the case where \( k = 2 \). For the following discussion, see [2, p.55]. Let \( \Lambda^2 \mathbb{R}^4 \) denote the space of skew-symmetric 2-tensors. The Grassmann manifold \( G(4,2) \) is naturally identified with the set of decomposable elements of norm one in \( \Lambda^2 \mathbb{R}^4 \). We have the Hodge operator \( * \) from \( \Lambda^2 \mathbb{R}^4 \) onto itself. The space \( \Lambda^2 \mathbb{R}^4 \) is decomposed to two orthogonal subsets \( E_1 \) and \( E_{-1} \) associated to the eigenvalue 1 and -1 of \( * \).
let $S^2_1$ and $S^2_{-1}$ be the sphere in $E_1$ and $E_{-1}$ of radius $1/\sqrt{2}$. Then $G(4,2)$ is equal to the product $S^2_1 \times S^2_{-1}$. Define a bilinear map $\zeta: \wedge^2 \mathbb{R}^4 \times \wedge^2 \mathbb{R}^4 \to \mathbb{R}$ by $\zeta(a, b) = || a \wedge b ||$, where $|| \cdot ||$ is the norm on $\wedge^2 \mathbb{R}^4 \cong \mathbb{R}$. Two planes $P_1$ and $P_2$ in $G(4,2)$ are transversal if and only if $\zeta(P_1, P_2) = 0$. Represent $P_1$ and $P_2$ by $(x_1, x_2)$ and $(y_1, y_2)$, where $x_1, y_1 \in S^2_1$ and $x_2, y_2 \in S^2_{-1}$. Then we have
\[ \zeta(P_1, P_2) = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle, \]
where $\langle \cdot, \cdot \rangle$ is the inner product of the vector space $E_1$ or $E_{-1}$.

For a smooth map $\theta: S^2 \to G(4,2)$, we define a smooth function $Z(\theta)$ on $S^2$ by $Z(\theta)(x) = \zeta(\theta(x), \theta(x'))$, by fixing $x'$ in $S^2$. Thus the principal $S^1$-bundle $\pi: S^3 \to S^2$ induced by an embedding $\theta: S^2 \to G(4,2)$ is a fibration by great circles if $Z(\theta)(x) = 0$ only when $x = x'$. Obviously $Z(\theta)(x) = 0$ at $x = x'$. We have:

Lemma 14. For a smooth map $\theta: S^2 \to G(4,2)$, the function $Z(\theta)$, for fixed $x' \in S^2$, is critical at $x = x'$.

Proof. Fix $P_2$ in $G(4,2)$. The function $\zeta(P_1, P_2)$ on $G(4,2)$ is critical at $P_1 = P_2$. Thus $Z(\theta)$ is also critical at $x = x'$.

Now consider the Hopf fibration $\pi_0: S^3 \to S^2$. 
The associated map \( \vartheta_0 : S^2 \to G(4,2) = S^2_1 \times S^2_{-1} \)
is given by \( \vartheta_0(x) = (1/\sqrt{2} x, \alpha_0) \), where \( \alpha_0 = (1/\sqrt{2}, 0, 0) \). For two points \( x = (x_1, x_2, x_3) \)
and \( x' = (x'_1, x'_2, x'_3) \) in \( S^2 \), we have

\[
\zeta(\vartheta_0(x), \vartheta_0(x')) = \langle x, x' \rangle - \frac{1}{2}
\]
\[
= -\frac{1}{2} \sum (x_i - x'_i)^2.
\]

Thus the function \( Z(\vartheta_0) \) is critical if and only if \( x = x' \). The symmetric matrix \( \sum_i \frac{\partial^2 Z(\vartheta_0)}{\partial x_i \partial x_j} \)
is positive definite.

Let \( \text{Emb}(S^2, G(4,2)) \) denote the set of smooth embeddings of \( S^2 \) in \( G(4,2) \) with \( C^2 \)-topology. Since
\( S^2 \) is compact, we obtain the following.

**Proposition 15.** There exists a neighborhood \( U \) of \( \vartheta_0 \)
in \( \text{Emb}(S^2, G(4,2)) \) such that the function \( Z(\theta)(x,x') \)
\( = \zeta(\theta(x), \theta(x')) \) is equal to zero if and only if \( x = x' \),
for any \( x, x' \in S^2 \) and \( \theta \in U \).

**Corollary 16.** In each direction in \( \text{Emb}(S^2, G(4,2)) \),
there is a deformation of fibrations of \( S^3 \) by great circles
starting from the Hopf fibration.
The group of diffeomorphisms of $S^2$, denoted by $\text{Diff } S^2$, acts naturally on $\text{Emb}(S^2, G(4,2))$. We denote by $\text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))$ the quotient space. Let $\pi : S^3 \to B$ be a fibration of $S^3$ by great circles. The $B$ is diffeomorphic to $S^2$. Thus we have the class $\{0\}$ in $\text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))$.

Let $\pi_1$ and $\pi_2$ be two fibrations of $S^3$ by great circles, and let $\{\theta_1\}, \{\theta_2\} \subset \text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))$ be the associated classes. We say that $\pi_1$ and $\pi_2$ are isometric if there exists a bundle map $F$ from $\pi_1$ to $\pi_2$ such that $F$ is an isometry of $S^3$ onto itself.

The group $O(4)$ acts naturally on $G(4,2)$ and on $\text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))$. We denote by $\text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))/O(4)$ the quotient space.

Proposition 17. Two fibrations $\pi_1$ and $\pi_2$ of $S^3$ by great circles are isometric if and only if the classes $\{\theta_1\}$ and $\{\theta_2\}$ in $\text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))/O(4)$ are equal.

Remark that we can choose the neighborhood $U$ in Proposition 15 such that $U$ is invariant by the actions of $\text{Diff } S^2$ and $O(4)$. The space $\text{Diff } S^2 \setminus U / O(4)$ is of infinite "dimension".
REFERENCES


