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On topological Blaschke conjecture I

Cohomological complex projective spaces

(Hajime SATO)

By a Blaschke manifold, we mean a Riemannian manifold \((M,g)\) such that, for any point \(m \in M\), the tangential cut locus \(C_m\) of \(m\) in \(T_mM\) is isometric to the sphere of constant radius. There are some equivalent definitions (see Besse[2, 5.43]). The Blaschke conjecture is that any Blaschke manifold is isometric to a compact rank one symmetric space. If the integral cohomology ring of \(M\) is equal to the sphere \(S^k\), or the real projective space \(RP^k\), this conjecture is proved by Berger with other mathematiciens [2, Appendix D]). We consider the case where the cohomology ring of \(M\) is equal to that of the complex projective space \(CP^k\).

We obtain the following theorem.

Theorem. Let \((M,g)\) be a \(2k\)-dimensional Blaschke manifold such that the integral cohomology ring is equal to that of \(CP^k\). Then \(M\) is PL-homeomorphic to \(CP^k\) for any \(k\).

Blaschke manifolds with other cohomology rings will be treated in subsequent papers.
If $(M,g)$ is a Blaschke manifold and $m \in M$, Allamigeon [1] has shown that the cut locus $C(m)$ of $m$ in $M$ is the base manifold of a fibration of the tangential cut locus $C_m$ by great spheres. We study the base manifold of such fibration by great circles. We apply the Browder-Novikov-Sullivan's theory in the classification of homotopy equivalent manifolds (see Wall[4]). Calculation of normal invariants gives our theorem. In Appendix, we give examples of non-trivial fibrations of $S^3$ by great circles. The author thanks to M.Mizutani and K.Masuda for the discussion of results in Appendix.

Detailed proof will appear elsewhere.
§1. Projectable bundles

In the paper [3], we have obtained a method of a calculation of the tangent bundle of the base space of an $S^1$-principal bundle. We will briefly recall that.

Let $X$ be a smooth manifold and let $\pi : L \rightarrow X$ be the projection of an $S^1$-principal bundle.

**Definition.** A vector bundle $p : E \rightarrow L$ over $L$ is projectable onto $X$, if there exists a vector bundle $\hat{p} : \hat{E} \rightarrow X$ over $X$ such that $\pi^* \hat{E} = E$. The map $\pi$ induces the bundle map $\pi_! : E \rightarrow \hat{E}$, which we call the projection. The bundle $\hat{E}$ is called the projected bundle.

Let $x$ be a point in $X$. For any $a, b \in \pi^{-1}(x) = S^1$, we have a linear isomorphism

$$\phi_{ab} : p^{-1}(a) \rightarrow p^{-1}(b)$$

of vector spaces defined by $\phi_{ab}(u) = v$, where $\pi_!(u) = \pi_!(v)$.

Then we have, for $a, b, c \in \pi^{-1}(x)$,

$$(1) \quad \phi_{bc} \phi_{ab} = \phi_{ac} .$$

Let $\pi^*L = \{(a, b) \in L \times L, \pi(a) = \pi(b)\}$ be the induced $S^1$-bundle over $L$ from $L$. We have two projections $\pi_1$,

$$\pi_2 : \pi^*L \rightarrow L$$

defined by $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

Let $\pi_i^*E$ $(i = 1, 2)$ be the induced vector bundle. The map $\phi : \pi^*L \rightarrow \text{Iso}(\pi_1^*E, \pi_2^*E)$ defined by $\phi(a, b) = \phi_{ab}$ is a continuous cross section of the bundle $\text{Iso}(\pi_1^*E, \pi_2^*E)$ over $\pi^*L$.
We call $\phi$ the projecting isomorphism associated with the projectable bundle $E$.

Proposition 1. Suppose given a vector bundle $E$ over $L$ and a cross section $\phi$ of the bundle $\text{Iso}(\pi_1^*E, \pi_2^*E)$ satisfying (1). Then we have a vector bundle $\hat{E}$ over $X$ such that $\pi^*\hat{E} = E$ and the projecting isomorphism is equal to $\phi$.

Now let $TL$ and $TX$ be the tangent bundles of $L$ and $X$ respectively. Let $\rho : S^1 \times L \to L$ be the free $S^1$-action. For each $t \in S^1$, the diffeomorphism $\rho(t) = \rho(t, \cdot)$ induces a bundle isomorphism $\rho(t)_* : TL \to TL$.

Proposition 2. The collection $\bigcup_{t \in S^1} \rho(t)_*$ induces a projecting isomorphism on the bundle $TL$ such that the projected bundle $TL$ is isomorphic to $TX \oplus 1$.

Proof. Choose a bundle metric on $TL$. Let $TL_1$ be the subbundle of $TL$ consisting of tangent vectors normal to the $S^1$-action. Then $TL_1$ is projected to $TX$. The line bundle tangent to the $S^1$-action is projected to the trivial line bundle on $X$. 

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§2. Pontrjagin classes

Let $S^{2k-1}$ be the unit sphere in $\mathbb{R}^{2k}$ and let $\pi : S^{2k-1} \longrightarrow B$ be a fibration of $S^{2k-1}$ by great circles. Thus, for each $b \in B$, $\pi^{-1}(b)$ is the intersection of $S^{2k-1}$ with a 2-plane in $\mathbb{R}^{2k}$. We write the 2-plane by $P(b)$. Let $\rho : S^1 \times S^{2k-1} \longrightarrow S^{2k-1}$ denote the free $S^1$-action.

Let $V(2k,2)$ and $G(2k,2)$, respectively, be the Stiefel and the Grassmann manifold consisting of orthogonal 2-frames or oriented 2-planes in $\mathbb{R}^{2k}$. Then the natural mapping $\lambda : V(2k,2) \longrightarrow G(2k,2)$ defines a principal $S^1$-bundle.

The mapping $\theta : B \longrightarrow G(2k,2)$ defined by $\theta(b) = P(b)$ is a smooth embedding. Let $\theta^*(\lambda)$ denote the induced bundle of $\lambda$ by $\theta$. Since $\pi$ is also the induced bundle of $\lambda$ by $\theta$, there exists a natural bundle isomorphism between $\pi$ and $\theta^*(\lambda)$ inducing the identity on $B$. Thus we obtain;

Lemma 3. We may suppose that the free $S^1$-action $\rho$ on $S^{2k-1}$ is equal to the restriction on $\pi^{-1}(b)$ of the linear action on $P(b)$ for every $b \in B$.

In the following, we always assume that $\rho$ is the linear action on each fibre. For each $x \in S^{2k-1}$, let $Kx$ denote the point $\rho(\frac{1}{4})x$ in $S^{2k-1}$, where we identify
$S^1$ with $[0, 1]/[0] \sim [1]$. Define a mapping $\mathcal{V} : S^{2k-1} \to \mathbb{V}(2k, 2)$ by $\mathcal{V}(x) = (x, Kx)$. This is a smooth embedding and is a bundle map inducing $\theta$ on the base manifolds. For an orthogonal 2-frame $w = (x, y)$, let $\tilde{\psi}(w)$ denote the vector $(x/\sqrt{2}, y/\sqrt{2})$ in $\mathbb{R}^{2k} \oplus \mathbb{R}^{2k}$. Then the map $\tilde{\psi} : \mathbb{V}(2k, 2) \to \mathbb{R}^{4k}$ is a smooth embedding of $\mathbb{V}(2k, 2)$ in $S^{4k-1} \subset \mathbb{R}^{4k}$. We identify $\mathbb{R}^{2k} \oplus \mathbb{R}^{2k}$ with $\mathbb{C}^{2k}$ such that the first summand $\mathbb{R}^{2k}$ is the real part and the second pure imaginary. On $\mathbb{C}^{2k} - 0$, we have the free action $\rho_0$ of $S^1$ as the multiplication by the complex number of norm one. Then $\tilde{\psi}$ is $S^1$-equivalent and we write by $\psi$ the induced map $\psi : G(2k, 2) \to \mathbb{C}P^{2k-1}$.

Let $\tilde{\mathcal{F}} : S^{2k-1} \to S^{4k-1}$ be the composition $\tilde{\mathcal{F}} = \psi \circ \mathcal{V}$ and $f = \psi \circ \theta : B \to \mathbb{C}^{2k-1}$. The map $\tilde{\mathcal{F}}$ is given by $\tilde{\mathcal{F}}(x) = (x/\sqrt{2}, Kx/\sqrt{2})$ for $x \in S^{2k-1}$.

We define a map $\tilde{F} : \mathbb{R}^{2k} - 0 \to \mathbb{C}^{2k} - 0$ by $\tilde{F}(tx) = t\tilde{\mathcal{F}}(x)$ for $t > 0$ and $x \in S^{2k-1}$. The map $\tilde{F}$ is a smooth embedding. Let $E$ denote the restriction of the tangent bundle $T(\mathbb{R}^{2k} - 0)$ of $\mathbb{R}^{2k} - 0$ on $S^{2k-1}$, and we write $p$ for the projection $E \to S^{2k-1}$. Then $\tilde{F}$ induces an injective bundle map $\tilde{F}_* : E \to \tilde{F}_*(E) \subset T(\mathbb{C}^{2k} - 0)|_{\tilde{F}(S^{2k-1})}$.

Now define a map $\tilde{G} : \mathbb{R}^{2k} - 0 \to \mathbb{C}^{2k} - 0$ by $\tilde{G}(tx) = (tx/\sqrt{2}, -tK/\sqrt{2})$ for $t > 0$ and $x \in S^{2k-1}$.
Then $\tilde{G}$ is also an embedding and $\tilde{G}$ induces an injective bundle map

$$G_* : E \rightarrow \tilde{G}_*(E) \subset T(\mathbb{C}^{2k-0} - 0) \mid \tilde{G}(S^{2k-1}) .$$

If $\tilde{\rho}_0$ denote the conjugate action of $S^1$ on $\mathbb{C}^{2k-0}$, then $G$ is $S^1$-equivariant concerning to this conjugate action.

For any $y \in \mathbb{C}^{2k}$, we naturally identify the tangent space $T_y \mathbb{C}^{2k}$ with $\mathbb{C}^{2k}$ itself. For $x \in S^{2k-1}$, let $E_x$ denote the fiber $p^{-1}(x)$. Then $\tilde{F}_*(E_x)$ and $\tilde{G}_*(E_x)$ are subvector spaces of $\mathbb{C}^{2k}$.

Since $K : S^{2k-1} \rightarrow S^{2k-1}$ is a diffeomorphism, we obtain:

Lemma 4. The vector spaces $\tilde{F}_*(E_x)$ and $\tilde{G}_*(E_x)$ are transversal. Thus they span $\mathbb{C}^{2k}$.

Let $T$ denote the restriction of the tangent bundle $T(\mathbb{C}^{2k})$ on $\tilde{F}(S^{2k-1})$. Then we have the direct sum decomposition by trivial vector bundles

$$T = \tilde{F}_*(E) \oplus \tilde{G}_*(E) .$$

Notice that $\tilde{G}_*(E)$ on $\tilde{G}(S^{2k-1})$ is identified with the subbundle in $T$ over $\tilde{F}(S^{2k-1})$ by an orientation reversing diffeomorphism of $S^{2k-1}$.

For any $t \in S^1$, we have the induced bundle isomorphisms $\rho_*(t) : E \rightarrow E$ and $\rho_{0*}(t) : T \rightarrow T$. 

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Lemma 5. The isomorphism $\rho(t) \ast 0$ is equal to the direct sum

\[ \rho_\ast(t) + \rho_\ast(t) . \]

By Proposition 1, we obtain that the projected bundle $T$, defined by the projecting isomorphism $\rho_\ast(t)$, is isomorphic to the Whitney sum;

\[ \hat{T} \cong \hat{E} \oplus \hat{E} . \]

On the other hand, by Proposition 2, we obtain the following.

Lemma 6. The bundle $\hat{T}$ has the complex structure. As a complex vector bundle, $\hat{T}$ is isomorphic to the Whitney sum $\hat{T}(\mathbb{C}P^{2k-1})|_{f(B)} \oplus 1$.

Lemma 7. As a real vector bundle, $\hat{E}$ is isomorphic to the bundle $T(B) \oplus 2$.

Consequently, we obtain that

\[ T(B) \oplus T(B) \oplus 4 \cong (T(\mathbb{C}P^{2k-1})|_{f(B)} \oplus 1)_\mathbb{R} . \]

Since the cohomology groups $H^*(B;\mathbb{Z})$ has no torsion element, by the product formula of Pontrjagin classes, we obtain the following.

Proposition 8. The Pontrjagin classes of the smooth manifold $B$ is equal to that of $\mathbb{C}P^{k-1}$, for any $k$. 

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§3. $Z_2$-invariants and proof of Theorem

Let $\mathcal{S}(\mathbb{CP}^{k-1})$ denote the set of PL-homeomorphism classes of closed PL-manifolds homotopy equivalent to $\mathbb{CP}^{k-1}$. The following results are due to Sullivan (cf. [4, §14 C])

Suppose that $k > 3$.

Proposition 9. For any $N \in \mathcal{S}(\mathbb{CP}^{k-1})$, there are invariants $s_{4i+2}(N) \in Z_2$ and $s_{4j}(N) \in Z$, for all integers $i, j$ satisfying $6 \leq 4i+2 < 2(k-1)$, $4 \leq 4j < 2(k-1)$. The invariants define a bijection of $\mathcal{S}(\mathbb{CP}^{k-1})$ with

$$( \bigoplus_{i} Z_2 ) \bigoplus ( \bigoplus_{j} Z )$$

The invariants $s_{4j}$ satisfy the following relations.

Proposition 10. If all the Pontrjagin classes of $N$ in $\mathcal{S}(\mathbb{CP}^{k-1})$ coincide with that of $\mathbb{CP}^{k-1}$, then $s_{4j}(N) = 0$ for all $j$.

Concerning $Z_2$-invariants $s_{4i+2}$, the following holds. Let $\mathcal{S}(\mathbb{RP}^{2k-1})$ denote the set of PL-homeomorphism classes of closed PL-manifolds homotopy equivalent to $\mathbb{RP}^{2k-1}$. This set is known to be equal to the isomorphism classes of homotopy triangulations of $\mathbb{RP}^{2k-1}$. Any $N \in \mathcal{S}(\mathbb{CP}^{k-1})$ is the base manifold of a PL free $S^1$-action on $S^{2k-1}$. By restricting the action to $Z_2 = S^0 \subset S^1$, we obtain a manifold homotopy equivalent to $\mathbb{RP}^{2k-1}$.
This defines a map

\[ \pi^b : \mathcal{S} (\mathbb{C}P^{k-1}) \rightarrow \mathcal{S} (\mathbb{R}P^{2k-1}) . \]

The following holds ([4, §14D.3]).

Proposition 11. Let \( N \) be an element in \( \mathcal{S} (\mathbb{C}P^{k-1}) \) such that \( \pi^b(N) \) is PL-homeomorphic to \( \mathbb{R}P^{2k-1} \). Then

\[ s_{4i+2}(N) = 0 , \]

for all \( i \).

Now let \( B \in \mathcal{S} (\mathbb{C}P^{k-1}) \) be the base manifold of the fibration of \( S^{2k-1} \) by great circles. Then, obviously, the image \( \pi^b(B) \in \mathcal{S} (\mathbb{R}P^{2k-1}) \) is PL-homeomorphic to \( \mathbb{R}P^{2k-1} \).

Combining the result of §2 with Propositions, we obtain ;

Proposition 12. The base manifold \( B \) of a fibration of \( S^{2k-1} \) by great circles is PL-homeomorphic to \( \mathbb{C}P^{k-1} \) if \( k \neq 3 \).

Now let us prove Theorem. Since the integral cohomology ring of \( M \) is equal to that of \( \mathbb{C}P^k \), \( M \) is simply connected ([2, 7.23]). Thus \( M \) is homotopy equivalent to \( \mathbb{C}P^k \). By Allamigeon's theorem, we know that \( M \) is PL-homeomorphic to the union of the disc \( D^{2k} \) with the \( D^2 \)-bundle associated with the fibration of \( S^{2k-1} \) by great circles. We write \( B \) for the base manifold of the fibration. If \( k = 3 \), by Proposition 9, \( M \) is
PL-homeomorphic to $\mathbb{CP}^3$ if and only if $s_4(M) = 0$.

The invariant $s_4(M)$ is calculated from the first Pontrjagin class $p_1(\mathbb{B})$ of $\mathbb{B}$. By Proposition 8 of §2, $p_1(\mathbb{B})$ is equal to $p_1(\mathbb{CP}^2)$. Thus we have $s_4(M) = 0$ and $M$ is PL-homeomorphic to $\mathbb{CP}^3$. Now suppose that $k \neq 3$. According to Proposition 12, $\mathbb{B}$ is PL-homeomorphic to $\mathbb{CP}^{k-1}$. The Euler class of the $S^1$-bundle is equal to a generator of $H^2(\mathbb{CP}^{k-1}; \mathbb{Z}) = \mathbb{Z}$. Thus the total space of the $D^2$-bundle is PL-homeomorphic to the tubular neighborhood of $\mathbb{CP}^{k-1}$ in $\mathbb{CP}^k$. Any orientation preserving PL-homeomorphism of $S^{2k-1}$ is isotopic to the identity. The attached manifold $M$ is PL-homeomorphic to $\mathbb{CP}^k$, which completes the proof of Theorem.
§4. Appendix

If $\pi : S^{2k-1} \to B$ is a fibration by great circles, we get the embedding $\vartheta : B \to G(2k,2)$. Since the planes $\vartheta(b)$ for all $b \in B$ give a foliation of $S^{2k-1}$, we have the following property.

(*) For two different points $b$ and $b'$ in $B$, the planes $\vartheta(b)$ and $\vartheta(b')$ are transverse.

The converse holds.

Lemma 13. Let $\pi : S^{2k-1} \to B$ be a principal $S^1$-bundle induced from the $S^1$-bundle $\lambda : W(2k,2) \to G(2k,2)$ by a smooth embedding $\vartheta : B \to G(2k,2)$. Suppose that, for any different points $b$ and $b'$ in $B$, the planes $\vartheta(b)$ and $\vartheta(b')$ are transversal. Then the bundle $\pi$ is a fibration of $S^{2k-1}$ by great circles.

Proof. Consider the union $\bigcup_b (\vartheta(b) \cap S^{2k-1})$. Then it covers $S^{2k-1}$ and give a fibration by great circles.

Now we consider the case where $k = 2$. For the following discussion, see [2, p.55]. Let $\Lambda^2 \mathbb{R}^4$ denote the space of skew-symmetric 2-tensors. The Grassmann manifold $G(4,2)$ is naturally identified with the set of decomposable elements of norm one in $\Lambda^2 \mathbb{R}^4$. We have the Hodge operator $*$ from $\Lambda^2 \mathbb{R}^4$ onto itself. The space $\Lambda^2 \mathbb{R}^4$ is decomposed to two orthogonal subsets $E_1$ and $E_{-1}$ associated to the eigenvalue 1 and -1 of $*$. 

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let \( S^2_1 \) and \( S^2_{-1} \) be the sphere in \( E_1 \) and \( E_{-1} \) of radius \( 1/\sqrt{2} \). Then \( G(4,2) \) is equal to the product \( S^2_1 \times S^2_{-1} \). Define a bilinear map \( \zeta : \wedge^2 \mathbb{R}^4 \times \wedge^2 \mathbb{R}^4 \rightarrow \mathbb{R} \) by \( \zeta(a,b) = \| a \wedge b \| \), where \( \| \cdot \| \) is the norm on \( \wedge^2 \mathbb{R}^4 \cong \mathbb{R} \). Two planes \( P_1 \) and \( P_2 \) in \( G(4,2) \) are transversal if and only if \( \zeta(P_1, P_2) = 0 \). Represent \( P_1 \) and \( P_2 \) by \((x_1, x_2)\) and \((y_1, y_2)\), where \( x_1, y_1 \in S^2_1 \) and \( x_2, y_2 \in S^2_{-1} \). Then we have

\[
\zeta(P_1, P_2) = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle,
\]

where \( \langle \ , \ \rangle \) is the inner product of the vector space \( E_1 \) or \( E_{-1} \).

For a smooth map \( \theta : S^2 \rightarrow G(4,2) \), we define a smooth function \( Z(\theta) \) on \( S^2 \) by \( Z(\theta)(x) = \zeta(\theta(x), \theta(x')) \), by fixing \( x' \) in \( S^2 \). Thus the principal \( S^1 \)-bundle \( \pi : S^3 \rightarrow S^2 \) induced by an embedding \( \theta : S^2 \rightarrow G(4,2) \) is a fibration by great circles if \( Z(\theta)(x) = 0 \) only when \( x = x' \). Obviously \( Z(\theta)(x) = 0 \) at \( x = x' \). We have;

**Lemma 14.** For a smooth map \( \theta : S^2 \rightarrow G(4,2) \), the function \( Z(\theta) \), for fixed \( x' \in S^2 \), is critical at \( x = x' \).

**Proof.** Fix \( P_2 \) in \( G(4,2) \). The function \( \zeta(P_1, P_2) \) on \( G(4,2) \) is critical at \( P_1 = P_2 \). Thus \( Z(\theta) \) is also critical at \( x = x' \).

Now consider the Hopf fibration \( \pi_0 : S^3 \rightarrow S^2 \).
The associated map $\theta_0 : S^2 \rightarrow G(4,2) = S^1 \times S^1$ is given by $\theta_0(x) = (1/\sqrt{2}, x, \alpha_0)$, where $\alpha_0 = (1/\sqrt{2}, 0, 0)$. For two points $x = (x_1, x_2, x_3)$ and $x' = (x'_1, x'_2, x'_3)$ in $S^2$, we have

$$\zeta(\theta_0(x), \theta_0(x')) = \langle x, x' \rangle - 1/2$$
$$= -1/2 \sum (x_i - x'_i)^2.$$

Thus the function $Z(\theta_0)$ is critical if and only if $x = x'$. The symmetric matrix $(\partial^2 Z(\theta_0) / \partial x_i \partial x_j)$ is positive definite.

Let $\text{Emb}(S^2, G(4,2))$ denote the set of smooth embeddings of $S^2$ in $G(4,2)$ with $C^2$-topology. Since $S^2$ is compact, we obtain the following.

**Proposition 15.** There exists a neighborhood $U$ of $\theta_0$ in $\text{Emb}(S^2, G(4,2))$ such that the function $Z(\theta)(x, x') = \zeta(\theta(x), \theta(x'))$ is equal to zero if and only if $x = x'$, for any $x, x' \in S^2$ and $\theta \in U$.

**Corollary 16.** In each direction in $\text{Emb}(S^2, G(4,2))$, there is a deformation of fibrations of $S^3$ by great circles starting from the Hopf fibration.
The group of diffeomorphisms of $S^2$, denoted by $\text{Diff } S^2$, acts naturally on $\text{Emb}(S^2, G(4,2))$. We denote by $\text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))$ the quotient space. Let $\pi : S^3 \rightarrow B$ be a fibration of $S^3$ by great circles. The $B$ is diffeomorphic to $S^2$. Thus we have the class $\{0\}$ in $\text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))$.

Let $\pi_1$ and $\pi_2$ be two fibrations of $S^3$ by great circles, and let $\{\theta_1\}, \{\theta_2\} \in \text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))$ be the associated classes. We say that $\pi_1$ and $\pi_2$ are isometric if there exists a bundle map $F$ from $\pi_1$ to $\pi_2$ such that $F$ is an isometry of $S^3$ onto itself.

The group $O(4)$ acts naturally on $G(4,2)$ and on $\text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))$. We denote by $\text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))/O(4)$ the quotient space.

Proposition 17. Two fibrations $\pi_1$ and $\pi_2$ of $S^3$ by great circles are isometric if and only if the classes $\{\theta_1\}$ and $\{\theta_2\}$ in $\text{Diff } S^2 \setminus \text{Emb}(S^2, G(4,2))/O(4)$ are equal.

Remark that we can choose the neighborhood $U$ in Proposition 15 such that $U$ is invariant by the actions of $\text{Diff } S^2$ and $O(4)$. The space $\text{Diff } S^2 \setminus U / O(4)$ is of infinite "dimension".
REFERENCES


