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Normal affine subalgebras of a polynomial ring

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Introduction. Let $R:=\mathbb{C}[x_1,\ldots,x_n]$ be a polynomial ring in n-variables over the complex number field $\mathbb{C}$. A cofinite subalgebra of R is a $\mathbb{C}$-subalgebra A of R such that R is an A-module of finite type. We consider exclusively a normal cofinite subalgebra A of R. The following results are known by far:

1. A is finitely generated over $\mathbb{C}$, and all invertible element of A are constants, i.e., $A^* = \mathbb{C}^*$.

2. Let X be the normal affine variety defined by A. Then $H_i(X;\mathbb{Z})$ is finite for all $i \geq 1$, X is simply connected and Pic is trivial (Gurjar [3] and Kumar [4]). Therefore A is factorial if X is nonsingular.

3. If $n = 2$ and A is regular, A is then a polynomial ring in two variables over $\mathbb{C}$ (Miyanishi [6]).

4. Suppose $n = 2$. Then X has at worst quotient singularitie (Brieskorn [1]). Moreover, if X is affine-ruled, i.e., X, by definition, contains a non-empty Zariski open set of the form $U_0 \times \mathbb{C}$, X has at worst cyclic quotient singularities (Miyanishi [6]).

We complement these results with the following:

THEOREM. *Let A be a normal cofinite subalgebra of $\mathbb{C}[x_1,x_2]$ and let $X:=\operatorname{Spec} A$. Then either $X \cong \mathbb{C}^2$ or $X \cong \mathbb{C}^2/G$, where G is a small finite subgroup of $GL(2,\mathbb{C})$. If A is factorial, X is isomorphic to a hypersurface in $\mathbb{C}^3$ defined by $x_1^2+x_2^3+x_3^5 = 0$.*

The theorem holds true even if we replace the ground field $\mathbb{C}$

by an algebraically closed field of characteristic $p \neq 2,3,5$. Moreover, the result is viewed as a global version of the following result of Brieskorn [1]:

<u>Among two-dimensional normal singular analytic local rings, a factorial one is isomorphic to</u> $\mathbb{C}\{x,y,z\}/(x^2+y^3+z^5)$.

1. <u>Proof of Theorem: Nonsingular case.</u>

Let X be a nonsingular algebraic surface defined over $\mathbb{C}$. Then there exists an open immersion of X into a nonsingular projective surface V such that $D := V-X$ consists of nonsingular curves which cross each other normally. Let K_V be the canonical divisor and denote by the same letter D the reduced effective divisor such that $\text{Supp } D = V-X$. Then we say that X has (logarithmic) Kodaira dimension $\kappa(X) = -\infty$ if $|n(D+K_V)| = \phi$ for every $n > 0$. Then the property $\kappa(X) = -\infty$ is independent of the choice of an immersion $X \hookrightarrow V$.

In proving the theorem, the following characterization of $\mathbb{C}^2$ plays a crucial role:

<u>Let</u> $X = \text{Spec } A$ <u>be a two-dimensional affine surface defined over</u> $\mathbb{C}$. <u>Then</u> $X \cong \mathbb{C}^2$ <u>if and only if the following three conditions are satisfied:</u>

(i) A <u>is factorial</u>, (ii) $A^* = \mathbb{C}^*$, (iii) X <u>is affine-ruled</u>.

<u>When</u> X <u>is nonsingular, the condition</u> (iii) <u>is equivalent to</u>

(iii)' $\kappa(X) = -\infty$.

(See Miyanishi [5; 6].)

Let X now be the same as in the theorem. Suppose X is nonsingular. Then $A := \Gamma(X, \underline{O}_X)$ is factorial by virtue of a result of Gurjar-Kumar, and $A^* = \mathbb{C}^*$ because A is a $\mathbb{C}$-subalgebra of

$R := \mathbb{C}[x_1, x_2]$. Moreover, since there is a finite morphism $\theta: \mathbb{C}^2 \longrightarrow X$, we have $\kappa(X) = -\infty$. Then $X \cong \mathbb{C}^2$ by virtue of the above-mentioned characterization of $\mathbb{C}^2$.

2. Proof of Theorem: Singular case.

We shall assume below that X is singular. Set

$X' := X - \text{Sing } X$, $S = \mathbb{C}^2$, $\theta : S \longrightarrow X$ the given finite morphism, $S' := \theta^{-1}(X')$, $\theta' := \theta|_{S'} : S' \longrightarrow X'$, $q' : Y' \longrightarrow X'$ the topological universal covering space of X'.

Then $\kappa(S') = -\infty$, and θ' factors as

$$\theta' : S' \xrightarrow{\pi'} Y' \xrightarrow{q'} X' .$$

Hence Y' is a nonsingular algebraic surface, and q' is a finit étale Galois covering with group G. Let

$A := \Gamma(X', \underline{O}_{X'}) = \Gamma(X, \underline{O}_X)$,

$B :=$ the integral closure of A in the function field $\mathbb{C}(Y')$,

$R := \mathbb{C}[x_1, x_2] = \Gamma(S, \underline{O}_S)$,

$Y :=$ Spec B,

$\pi: S \longrightarrow Y$ and $q : Y \longrightarrow X$: the finite morphisms induced by the canonical inclusions $B \subset R$ and $A \subset B$, respectively.

Then we know that:

(i) $A = R \cap \mathbb{C}(X)$ and $B = R \cap \mathbb{C}(Y')$;

(ii) Y is a normal affine surface defined over $\mathbb{C}$ such that Y' is an open set of Y with $\dim(Y-Y') \leq 0$;

(iii) $\theta = q \cdot \pi$, $\pi' = \pi|_{S'}$ and $q' = q|_{Y'}$;

(iv) G acts regularly on Y, and $X \cong Y/G$.

On the other hand, Y' is simply connected by the definition, Pic is a torsion group because S' is a finite covering of Y', and $\dim(Y-Y') \leqq 0$. Therefore $\mathrm{Pic}\, Y' = (0)$, and the divisor class group $C\ell(Y)$ is trivial, i.e., B is factorial. Since $B \subset R$, we have $B^* = \mathbb{C}^*$. Hence _if_ Y' _is affine-ruled_, so is Y, and $Y \cong \mathbb{C}^2$ by virtue of the characterization theorem of $\mathbb{C}^2$. The group G then becomes a finite subgroup of $\mathrm{Aut}\, \mathbb{C}^2 = \mathrm{Aut}\, \mathbb{C}[x_1, x_2]$, which is, up to conjugation, a finite subgroup of $GL(2,\mathbb{C})$. Let N be the normal subgroup of G consisting of all pseudo-reflections. Then $\mathbb{C}^2/N$ is isomorphic to $\mathbb{C}^2$, and $X \cong (Y/N)/(G/N) \cong \mathbb{C}^2/(G/N)$. Hence we may assume that G is _small_, i.e., G contains no pseudo-reflections.

Note that $\kappa(Y') = -\infty$ because S' is a finite covering of Y' and $\kappa(S') = -\infty$. We shall show that Y' is affine-ruled. By _reductio absurdum_, we assume that Y' is _not_ affine-ruled. Then we have the following:

THEOREM (Tsunoda-Miyanishi [8]). _There exist a Zariski open set_ U _of_ Y' _and a proper birational morphism_ $\phi : U \longrightarrow Z$ _from_ U _onto a nonsingular algebraic surface_ Z _defined over_ $\mathbb{C}$ _such that:_

(i) _Either_ $U = Y'$ _or_ $Y'-U$ _has pure dimension_ 1;

(ii) Z _is a Platonic_ $\mathbb{C}^*$-_fiber space_.

A nonsingular algebraic surface Z is called _a Platonic_ $\mathbb{C}^*$-_fiber space_ if there exists a surjective morphism $f : Z \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ such that general fibers of f are isomorphic to $\mathbb{C}^*$ and that f has exactly three singular fibers $\Delta_i = \mu_i\Gamma_i$ $(i = 0,1,2;\ \mu_0 \leqq \mu_1 \leqq \mu_2)$ with $\Gamma_i \cong \mathbb{C}^*$, where $\{\mu_0,\mu_1,\mu_2\} = \{2,2,n\}$ $(n \geqq 2)$, $\{2,3,3\}$, $\{2,3,4\}$ or $\{2,3,5\}$.

Since $U \subset Y' \subset Y$ and Y is affine, U does not contain any complete curve. Therefore $\underline{\phi} : U \longrightarrow Z$ is an isomorphism. We claim that $U = Y'$. Otherwise, since $Y'-U$ has pure dimension 1 and Pic $Y' = (0)$, there exists an element b of $B = \Gamma(Y', \underline{O}_{Y'})$ such that $\mathrm{Supp}\,(b)_{0,Y'} = \mathrm{Supp}(Y'-U)$. Hence b is invertible on U and $b \notin \mathbb{C}^*$. Meanwhile, there exists a completion W of U such that W is a normal projective surface, W has at worst quotient singularities and $W-U$ consists of two connected components, the one being a single irreducible curve and the other being a single quotient singular point (see Example 1 in the Section 3). Since $(b)_W$ has support on $W-U$, b must be a constant, i.e., $b \in \mathbb{C}^*$. This is a contradiction. Hence $U = Y'$. In order to complete the proof of the first assertion of the theorem we make use of the following:

THEOREM (Miyanishi [7]; Fujita [2]). Let $\tilde{U} \longrightarrow U$ be the topological universal covering space of U. Then $\tilde{U}$ is an affine-ruled nonsingular algebraic surface. Moreover we have

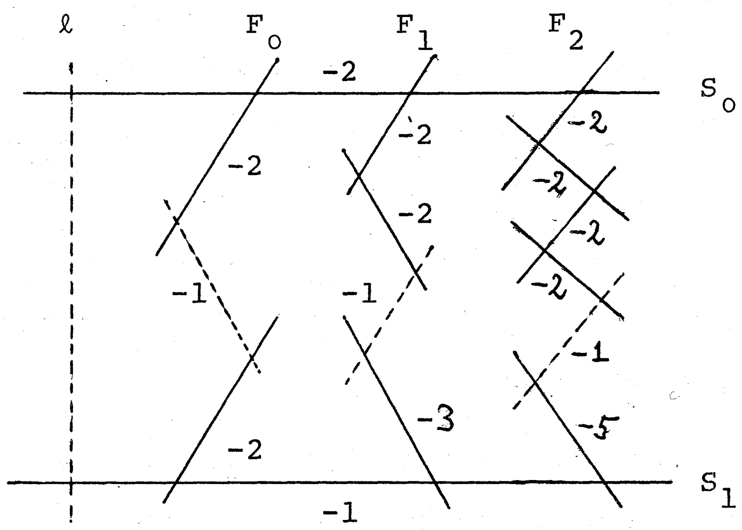
$$\pi_1(U) \cong \begin{cases} D_{2n} & \text{if } \{\mu_0,\mu_1,\mu_2\} = \{2,2,n\} \\ A_4 & \text{if } \{\mu_0,\mu_1,\mu_2\} = \{2,3,3\} \\ S_4 & \text{if } \{\mu_0,\mu_1,\mu_2\} = \{2,3,4\} \\ A_5 & \text{if } \{\mu_0,\mu_1,\mu_2\} = \{2,3,5\}\ , \end{cases}$$

where D_{2n} is a dihedral group of order $2n$, (see Example 2 of the Section 3).

However Y' is simply connected by the definition. This is apparently a contradiction. Thus Y' is affine-ruled, and we are done.

3. Examples.

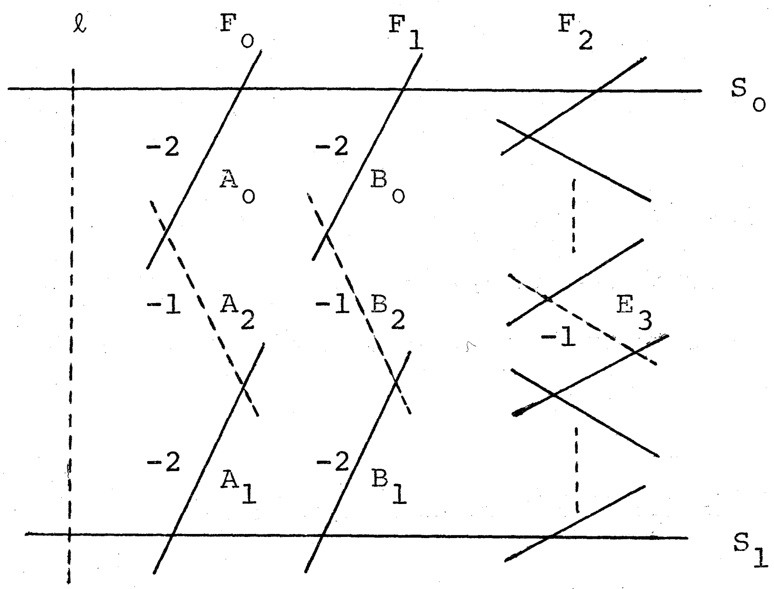
(1) Let T be a hypersurface $x_1^2+x_2^3+x_3^5 = 0$ in $\mathbb{C}^3$ and let T' be the minimal resolution of the unique singular point P:= (0,0,0) of T. Then T' is embedded into a nonsingular projective surface V in such a way that, in the configuration below, the top solid lines represent the exceptional curves arising from the minimal resolution of singularity at P, and the bottom solid lines represent the curves attached to T' to compactify the surface T':



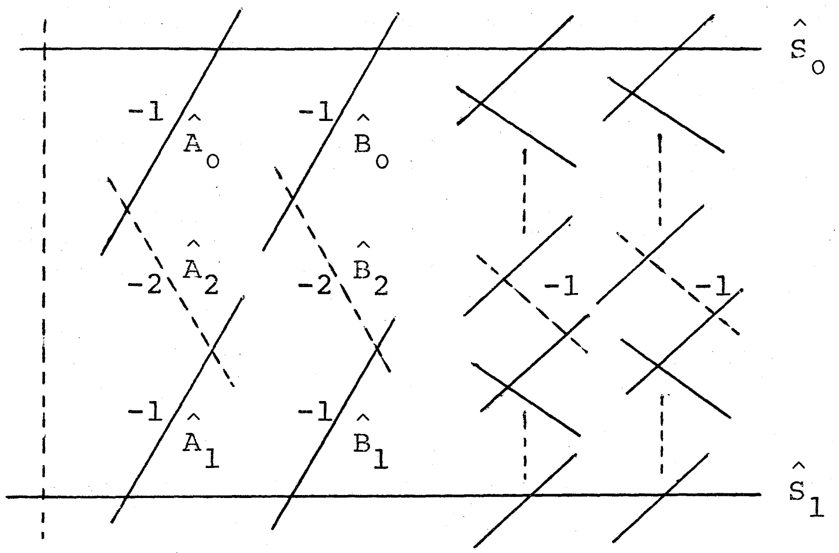
where "solid line" = a nonsingular rational curve, "broken line with weight -1" = an exceptional curve of the first kind and ℓ is a fiber of f.

Moreover V has a structure of a $\mathbf{P}^1$-fibration $f : V \longrightarrow C$ over $C \cong \mathbf{P}^1_{\mathbb{C}}$ such that f has three singular fibers F_i (i = 0,1,2) and two cross-sections S_0, S_1 as indicated in the configuration. Let U:= T-{P}. Then U $\cong$ V-(all solid lines), and U is a Platonic $\mathbb{C}$*-fiber space with the triple {2,3,5}. The top solid lines contract down back to the quotient singular point P, and the bottom solid lines (sprouting from S_1) contract down to three cyclic quotient singular points. With these contractions performed, we obtain a normal projective surface W such that T is an open set of W and $W-T$ consists of a single irreducible curve which is the proper transform of S_1.

(2) Consider a Platonic $\mathbb{C}^*$-fiber space U with the triple $\{2,2,n\}$ In general, it can be embedded into a nonsingular projective surface V, whose boundary $V-U$ has the configuration as shown in the following picture:



Moreover, V has a structure of a $\mathbf{P}^1$-fibration $f : V \longrightarrow C$ over $C \cong \mathbf{P}^1_{\mathbb{C}}$ for which S_0 and S_1 are cross-sections and F_0, F_1 and F_2 are the only singular fibers. Let D be the reduced effective divisor on V supported by all solid lines. Then $D+K_V \sim \ell - (A_2+B_2+E_3)$, where ℓ is a fiber of f. Hence $2(E_3+D+K_V) \sim A_0+A_1 +B_0+B_1$. This implies that there exists a finite double covering $\alpha : \hat{V} \longrightarrow V$ ramified only over $A_0+A_1+B_0+B_1$. The surface $\hat{V}$ has a $\mathbf{P}^1$-fibration $\hat{f} : \hat{V} \longrightarrow \hat{C} \cong \mathbf{P}^1_{\mathbb{C}}$ which is induced by the $\mathbf{P}^1$-fibration $f : V \longrightarrow C$ and has the configuration given in the next page. Indeed, $\hat{V}$ is the normalization of $V \times_C \hat{C}$, where $\hat{C} \longrightarrow C$ is the double covering ramified over the points $f(F_0)$ and $f(F_1)$.



Now contracting $\hat{A}_0$, $\hat{A}_1$, $\hat{B}_0$ and $\hat{B}_1$, we are reduced to the case where $\hat{f}$ has only two singular fibers of the same form as F_2. It is now a good exercise to show that $\hat{V}$ - (all solid lines) is affine-ruled.

4. Proof of Theorem: The second assertion.

We start with the following situation:

$G \subset GL(2,\mathbb{C})$: a small finite subgroup,

$X := \mathbb{C}^2/G$: a singular normal affine surface,

$P :=$ the unique singular point of X which is the image of the point of origin $(0,0)$ of $\mathbb{C}^2$,

$A := \Gamma(X, \underline{O}_X)$.

Then A is factorial if and only if $\mathrm{Pic}(X-\{P\}) = (0)$. A line bundle L on $X-\{P\}$ is constructed from a multiplicative character χ of G in the following way: Let L be a line bundle on $X-\{P\}$ and let $\theta : \mathbb{C}^2 \longrightarrow X$ be the (finite) quotient morphism. Then θ^*L is a trivial line bundle on $\mathbb{C}^2-\{0\}$. The action of G on θ^*L is given by

$$(x,t) \in (\mathbb{C}^2-\{0\}) \times \mathbb{C} \longmapsto ({}^g x, \chi(g,x)t) \in (\mathbb{C}^2-\{0\}) \times \mathbb{C},$$

where $g \in G$ and $\chi(g,x) \in \mathbb{C}^*$. Moreover, we have

$$\chi(gg';x) = \chi(g;{}^{g'}x)\chi(g';x) \quad \text{for } g, g' \in G.$$

If $g \in G$ is fixed, then we have

$$\chi(g;x) \in \Gamma(\mathbb{C}^2-\{o\},\underline{O}^*) = \mathbb{C}[x_1,x_2]^* = \mathbb{C}^*.$$

Hence $\chi(g,x)$ is independent of x. Write $\chi(g) = \chi(g;x)$. Then $\chi : G \longrightarrow \mathbb{C}^*$ is a multiplicative character. Conversely, a multiplicative character χ of G defines a line bundle $L_\chi :=$ $(\mathbb{C}^2-\{0\}) \times \mathbb{C}/G$ with respect to the action of G as described above. Thus we have a $1-1$ correspondence between

$$L \in \mathrm{Pic}(X-\{P\}) \longleftrightarrow \chi \in \hat{G}.$$

Then we have:

$$\begin{aligned} \mathrm{Pic}(X-\{P\}) = (0) &\Longleftrightarrow \hat{G} = (1) \\ &\Longleftrightarrow G \text{ is a binary icosahedral group in } SL(2,\mathbb{C}) \\ &\Longleftrightarrow X \text{ is isomorphic to a hypersurface } x_1^2+x_2^3+x_3^5 = 0 \text{ in } \mathbb{C}^3. \end{aligned}$$

This completes the proof of the theorem.

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