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Locally finite higher derivations and its applications

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§1. Definition.

Let A be a reduced ring with identity element. A higher derivation $\underline{D}$ is a system of additive endomorphism D_i of A ($i = 0,1,2,\cdots$, $D_0 = id$) such that they satisfy

(1) $D_n(ab) = \sum_{i=0}^{n} D_i(a)D_{n-i}(b) \qquad (n = 1,2,\cdots)$

for any element a, b of A. This is equivalent to saying that the mapping ϕ of A into a formal power series ring $A[[T]]$ defined by

(2) $\phi(a) = a + \sum_{i\geqq 1} D_i(a)T^i$

is a ring homomorphism of A into $A[[T]]$ such that $\varepsilon\phi = id$, where ε is an augmentation $A[[T]] \longrightarrow A$ defined by $\varepsilon(a) = a$ and $\varepsilon(T) = 0$. A higher derivation D of A is called iterative if they satisfy moreover the identity

(3) $D_iD_j = \binom{i+j}{i} D_{i+j}$.

A higher derivation D is called locally finite if $I_m(\phi)$ is contained in a polynomial ring $A[T]$.

Lemma 1. Let A be a reduced ring and let D be a locally finite higher derivation (ℓfhd). Then any unit in A is killed by D, i. e., if a is a unit of A then $D_i(a) = 0$ for $i \geqq 1$.

Corollary. The following rings have no non-trivial ℓfhd.

(1) A field: (2) A local ring (3) An integral domain with non-zero Jacobson radical.

§2. Application to invariance of rings.

A ring A is called a strongly invariant ring whenever the relation $A[X_1,\cdots,X_n] = B[Y_1,\cdots,Y_n]$ implies $A = B$ where $X_1,\cdots,X_n$ are indeterminates and $Y_1,\cdots,Y_n$ are independent variables over B.

A ring A is called an invariant ring whenever the relation $A[X_1,\cdots,X_n] = B[Y_1,\cdots,Y_n]$ implies A and B are isomorphic.

Theorem 2. Let A be a reduced ring, then A is strongly invariant if A has no non-trivial ℓfhd. A is not strongly invariant if A has a non-trivial ℓfihd.

Theorem 3. Let A be an affine domain over a field k such that $\mathrm{Spec}(A)(k)$ is dense in $\mathrm{Spec}(A)$. If A has a non-trivial ℓfhd, then $Q(A)$ is rational over k. If, moreover, A is normal A is isomorphic to a polynomial ring $k[t]$.

§3. An integral domain with non-trivial ℓfihds.

Let A be an integral integral domain and let D be a ℓfihd. Let A_i be defined by

$$A_i = \{a \in A \mid D_n(a) = 0 \quad \text{for all} \quad n > i\}.$$

An integer n such that $A_{n-1} \subsetneq A_n$ is called a jump index.

Proposition 4. Let A, D, A_i be as above. If the characteristic of an integral domain A is zero. Then the first jump index is 1. If the characteristic of A is a positive prime integer p, then we have the following:

(i) The first jump index is a power of p, say, $q = p^s$.

(ii) The m-th jump index is mp^s $(m = 1,2,\cdots)$.

(iii) Let a be an element of $A_q \setminus A_{q-1}$. Then $\mathrm{Supp}(a)$ consists of powers of p where $\mathrm{Supp}(a) = \{k \in \mathbb{N} \mid D_k(a) \neq 0\}$. Moreover if $k \in \mathrm{Supp}(a)$, then $D_k(a)$'s are D-constant, i. e., $D_k(a) \in A_0$.

Let a $(\in A)$ be an element of A. Then $\ell(a)$ is, by definition, the largest inteter m such $a \in A_{mq} \setminus A_{(m-1)q}$. In this case m is called the index of a and will be denoted by $i(a)$.

Let A be an integral domain. Let $\underline{D}^{(1)},\cdots,\underline{D}^{(n)}$ be n-ℓfhd of A. We say that $\underline{D}^{(1)},\cdots,\underline{D}^{(n)}$ are independent if

$$B_1 \cap \cdots \cap \hat{B}_i \cap \cdots \cap B_n \not\subset B_i \qquad (i = 1,\cdots,n)$$

where $B_i = (\underline{D}^{(i)})^{-1}(0) = \{a \in A \mid D_j^{(i)}(a) = 0 \quad \text{for} \quad j = 1,2,\cdots\}$. They are said to be commutative if we have

$$D_j^{(i)} D_\ell^{(k)} = D_\ell^{(k)} D_j^{(i)} \qquad (\text{for all} \quad j,\ \ell = 1,2,\cdots).$$

Theorem 5. Let A be an integral domain and let $\underline{D}^{(i)}$ $i = 1,2,\cdots,n$ be n-independent, mutually commutative, non-trivial ℓfihd, and let $B^i = (\underline{D}^{(i)})^{-1}(0)$. Then there exists an element ω in $A_0 = \bigcap_{i=1}^{n} B_i$, such that

$$A[\omega^{-1}] = A_0[\omega^{-1}][x_1,\cdots,x_n]$$

where $x_1,\cdots,x_n$ are independent variables over $Q(A_0)$.

Corollary 6. Let A be an affine domain of transcendence degree n over a field k such that $\bar{k} \cap A = k$. Assume that A has n-independent, mutually commutative, non-trivial ℓfihd $\underline{D}^i$ $(i = 1,\cdots,n)$. Then A is a polynomial ring in n-variables over k. Moreover if $x_1,\cdots,x_n$ are elements of A such that

(1) x_i is an element of index 1 with respect to $\underline{D}^i$.

(2) $x_i \in \bigcap_{j\neq i} (\underline{D}^j)^{-1}(0)$.

Then $A = k[x_1,x_2,\cdots,x_n]$.

Corollary 7. Let $F_1,\cdots,F_n$ be polynomials in n-variables $x_1,\cdots,x_n$ over a field k of characteristic zero. Assume

$$\frac{\partial(F_1,\cdots,F_n)}{\partial(x_1,\cdots,x_n)} \in k^*.$$

If the derivation D_i defined by

$$D_i(g) = \frac{\partial(F_1,\cdots,\overset{i}{\smile}{g},\cdots,F_n)}{\partial(x_1,\cdots,x_i,\cdots,x_n)}, \quad (i = 1,2,\cdots,n)$$

is locally nilpotent for any i, i. e., any element is killed by some power of D_i, then we have $k[F_1,\cdots,F_n] = k[x_1,\cdots,x_n]$.

Corollary 8. Separable forms of affine n-spaces $(n = 1, 2)$ are trivial.

§4. Characterization of a polynomial ring $k[x,y]$.

Theorem 9. Let k be an algebraically closed field of arbitrary characteristic and let A be an integral domain satisfying conditions:

(1) There exists a non-trivial ℓfihd $\underline{\Delta}$ over k.

(2) The constant ring A_0 of $\underline{\Delta}$ is either (2,1) a principal ideal domain finitely generated over k or (2,2) a DVR whose residue field is k.

(3) Any prime element of A_0 remains prime in A.

Then A is a polynomial ring in one variable over A_0.

As application of Theorem 9 we have the following two theorems.

Theorem 10 (T. Kambayashi). Let (D,M) be a DVR with algebraically closed residue field $k = D/M$. Let K be the field of fractions of D. Let A be a flat D-algebra of finite type. Assume that $A \otimes_D K = K[t]$ and $A \otimes_D K$ is an integral domain. Then A is isomorphic to a polynomial ring $D[T]$.

Theorem 11. Let k be an algebraically closed field and let A be a normal affine domain over k such that

(i) $\dim A = 2$.

(ii) $A^* = k^*$ where $*$ denote the set of units.

(iii) Either A is UFD or $Q(A)$ is unirational.

Let Δ be a non-trivial ℓfihd of A over k. Then the constant ring A_0 of Δ is a polynomial ring over k.

Theorem 12 (M. Miyanishi). Let k be an algebraically closed field and let A be a finitely generated integral domain over k. Assume the following:

(i) $\dim A = 2$.

(ii) $A^* = k^*$.

(iii) A is UFD.

(iv) A has a non-trivial ℓfihd.

Then $A \cong k[x,y]$ where x, y are independent variables.

§5. Lines in an affine 2-space.

An affine plane curve C defined over k by the equation: $f(x,y) = 0$ is called a quasi-line if the coordinate ring $A = k[x,y]/(f)$ is isomorphic to a polynomial ring $k[t]$. C is called a line if there exists a curve Γ: $g(x,y) = 0$ such that $k[x,y] = k[f,g]$.

Theorem 13. Let k be an algebraically closed field and let $C : f(x,y) = 0$ be a curve defined over k. Then the following conditions are equivalent to each other

(1) C is a line.

(2) There exists a ℓfihd Δ such that $\Delta(f) = 0$.

(3) $C_u : f(x,y) - u = 0$ is a quasi-line over $k(u)$ where u is an indeterminate.

Theorem 14. Let k be an algebraically closed field of characteristic zero and let $C : f(x,y) = 0$ be an irreducible curve over k. Then C is a line if and only if the derivation

$$D_f = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}$$

is locally nilpotent.

References

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