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<td>Author(s)</td>
<td>Rees, David</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1983), 484: 22-30</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1983-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/103437">http://hdl.handle.net/2433/103437</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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GENERAL ELEMENTS OF IDEALS IN LOCAL RINGS

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In many situations arising in the theory of local rings, it is necessary to make use of elements $x_1, \ldots, x_s$ of ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ which are sufficiently general in some sense, depending on the particular situation involved. The purpose of this lecture is to describe a general set-up in which such general elements can be defined which satisfy the required conditions in most such situations and to give an illustration of its application.

We suppose that $(Q, \mathfrak{m}, k)$ is a local ring of dimension $d$. We first construct the general extension $Q_g$ of $Q$. Let $X_1, X_2, \ldots$ be a countable sequence of indeterminates over $Q$. Then $Q_g$ is the localisation of $Q[X_1, X_2, \ldots]$ at the prime ideal $\mathfrak{m}[X_1, X_2, \ldots]$. It follows from a general result of Grothendieck that $Q_g$ is noetherian (alternatively one can prove that if $\mathfrak{a}$ is a finitely generated ideal of $Q_g$, then $\bigcap_{n=1}^{\infty} (\mathfrak{a} + \mathfrak{m}_g^n) = \mathfrak{a}$, and then, observing that the completion of $Q_g$ is noetherian, use the above to show that if $\mathfrak{a}Q_g = \mathfrak{a}'Q_g$ where $\mathfrak{a}'$ is a finitely generated ideal of $Q_g$ contained in $\mathfrak{a}$, then $\mathfrak{a} = \mathfrak{a}'$.)

Now suppose that $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ are ideals of $Q$, and that $\mathfrak{a}_1$ has a basis $a_{i1}, \ldots, a_{im_i}$. Write $M_i = m_{i1} + \ldots + m_{im_i}$. Then we term $x_1, \ldots, x_s$ an independent set of general elements of $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ if there exists an automorphism $T$ of $Q_g$ over $Q$ such that
\[ T(x_i) = \sum_{j=1}^{m_i} x_{M_i-1+j}^{a_{ij}} \quad (i = 1, \ldots, s). \]

It is a simple matter to prove that this definition is independent of the choice of bases of \( \mathfrak{a} \sigma_1, \ldots, \mathfrak{a} \sigma_s \). It also follows that the ideal \((x_1, \ldots, x_s) \cap Q\) of \( Q \) and the \( Q \)-algebra \( \mathcal{O}_g/(x_1, \ldots, x_s) \) (to within isomorphism as a \( Q \)-algebra) depend only on the ideals \( \mathfrak{a} \sigma_1, \ldots, \mathfrak{a} \sigma_s \). I will only consider the first in the case when the ideals \( \mathfrak{a} \sigma_1, \ldots, \mathfrak{a} \sigma_s \) are all equal to \( \mathfrak{a} \). Let \( a(\mathfrak{a}) \) denote the analytic spread of \( \mathfrak{a} \), and \( v(\mathfrak{a} L) \) the minimal number of generators of \( \mathfrak{a} \).

i) if \( s < a(\mathfrak{a}) \), the ideal \((x_1, \ldots, x_s) \cap Q\) is nilpotent;

ii) if \( s = a(\mathfrak{a}) \), \((x_1, \ldots, x_s)\) is a reduction of \( \mathfrak{a} \sigma Q_g \) and hence \((x_1', \ldots, x_s') \supseteq \mathfrak{a} \sigma^n Q_g\) for \( n \) large, and hence \((x_1, \ldots, x_s) \cap Q\) contains a power of \( \mathfrak{a} \);

iii) if \( s \geq v(\mathfrak{a}) \), we have \((x_1, \ldots, x_s) \cap Q = \mathfrak{a} \).

Now we consider the second. In this case we will be concerned with the case when \( s = d-1 \) or \( d \), and the ideals \( \mathfrak{a} \sigma_1, \ldots, \mathfrak{a} \sigma_s \) are all \( \mathfrak{m} \mathfrak{w} \)-primary. Let \( N \) be any integer and define \( Q_N \) to be the ring \( Q[Y_1, \ldots, Y_N] \) localised at \( \mathfrak{m}[Y_1, \ldots, Y_N], Y_1, \ldots, Y_N \) being indeterminates over \( Q \). If we replace \( Y_i \) by \( X_i \), it is clear that we can consider \( Q_N \) as a subring of \( Q_g \). Now suppose that \( \mathfrak{a} \sigma \) is any ideal of \( Q_g \). Then for some \( N \), \( \mathfrak{a} \sigma \) is generated by elements of the sub-ring \( Q_N \) of \( Q_g \) and therefore \( \mathfrak{a} \sigma = (\mathfrak{a} \sigma \cap Q_N) Q_g \). Now we have an isomorphism of \( (Q_g)_N \to Q_g \) in which \( X_i \) maps to \( X_{N+i} \) and \( Y_i \to X_i \) for \( i = 1, \ldots, N \). It follows that \( Q_g/\mathfrak{a} \sigma \) is isomorphic to
$(Q_{g})_{N}/\alpha^{*}$, where $\alpha^{*}$ is an ideal of $(Q_{g})_{N}$ meeting $Q_{g}$ in $(Q \cap \alpha)Q_{g}$. The case that will concern us is when $\alpha$ is generated by general elements $x_{1}, \ldots, x_{d-1}$ of $\mathfrak{m}$-primary ideals $\alpha_{1}, \ldots, \alpha_{d-1}$ of $Q$.

For simplicity of exposition, we will restrict ourselves to the case when $Q$ is a domain. Then $Q_{g}/(x_{1}, \ldots, x_{d-1})$ is a local ring of dimension 1. Now suppose $y_{i}, z_{i}$ $(i = 1, \ldots, d-1)$ is a set of independent general elements of the ideals $\alpha_{1}, \alpha_{1}', \ldots, \alpha_{d-1}', \alpha_{d-1}$. Now choose $N$ so that the elements $y_{i}, z_{i}$ $(i = 1, \ldots, d-1)$ are all contained in the sub-ring $Q_{N}$ of $Q_{g}$. Then it is not difficult to prove that the elements $w_{i} = y_{i} - x_{N+1}z_{i}$ $(i = 1, \ldots, d-1)$ form a set of independent general elements of $\alpha_{1}, \ldots, \alpha_{d-1}'$. We further note that for each $i$, the elements $y_{i}, z_{1}, \ldots, z_{d-1}$ generate an $\mathfrak{m}Q_{g}$-primary ideal of $Q_{g}$. We now quote a general result which will be proved in an appendix:

Let $Q$ be a local domain of dimension $d$, and let $y_{i}, z_{i}$ $(i = 1, \ldots, d-1)$ be elements of $Q$ such that $y_{i}, z_{1}, \ldots, z_{d-1}$ generate an $\mathfrak{m}$-primary ideal for each $i$. Then, if $B$ is the ring $Q[y_{1}/z_{1}, \ldots, y_{d-1}/z_{d-1}]$, $B/\mathfrak{m}B$ is isomorphic to $k[X_{1}, \ldots, X_{d-1}]$, where $k = Q/\mathfrak{m}$, and $X_{1}, \ldots, X_{d-1}$ are indeterminates over $k$;

i) if $L$ denotes $B$ localised at the prime ideal $\mathfrak{m}[y_{1}/z_{1}, \ldots, y_{d-1}/z_{d-1}]$, and $Q(X)$ denotes the ring $Q[X_{1}, \ldots, X_{d-1}]$ localised at $\mathfrak{m}[X_{1}, \ldots, X_{d-1}]$, where $X_{1}, \ldots, X_{d-1}$ are indeterminates over $Q$, then the kernel of the homomorphism of $Q(X)$ onto $L$ in which $X_{i} \rightarrow y_{i}/z_{i}$ $(i = 1, \ldots, d-1)$ is a prime ideal $\mathfrak{P}$ containing the ideal $\mathfrak{I} = (y_{1}-z_{1}X_{1}, \ldots, y_{d-1}-z_{d-1}X_{d-1})$ and $\mathfrak{P}/\mathfrak{I}$ is annihilated by a power of $\mathfrak{m}$. 

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Applying this result, we see that, replacing \( Q \) by \( Q_g \) and giving \( y_i, z_i \) their original meaning, the ring \( L \) obtained in this situation is isomorphic to \( Q_g/(x_1, \ldots, x_{d-1}): m^n \) if \( n \) is large enough.

It follows that we can consider \( L \) in two ways, first as a homomorphic image of \( Q_g \), and second as a local ring containing \( Q_g \) and contained in its field of fractions \( F_g \). Further the maximal ideal of \( L \) is \( m_L \) and \( m_L \cap Q_g = m_Q_g \). Now \( L \) is 1-dimensional.

Hence, by the Krull-Akizuki theorem, the integral closure \( L^* \) of \( L \) in \( F_g \) is the intersection of a finite set of discrete valuation rings. Let the associated valuations be \( V_1, \ldots, V_q \) and let their restriction to the field of fractions \( F \) of \( Q \) be \( v_1, \ldots, v_q \). Then \( v_1, \ldots, v_q \) are independent of the choice of the elements \( y_i, z_i \).

Now we must digress to consider valuations on \( Q_g \). Suppose that \( V \) is a valuation \( > 0 \) on \( Q_g \), and \( > 0 \) on \( m_Q_g \), and taking integer values. If \( K_V \) is the residue field of \( V \), then \( K_V \) is an extension of \( k_g \), and an old result of Zariski states that \( \text{tr.deg}_k K_V \leq d-1 \). Now let \( v \) be the restriction of \( V \) to \( F \). Then it is quite easy to prove that

\[
\text{tr.deg}_k K_V \geq \text{tr.deg}_k K_v.
\]

Now I recall another old result; due in this case to Northcott. Let \( K \) denote the residue field of \( L \) (which is a pure transcendental extension of \( k_g \) of transcendence degree \( d-1 \)). Now the valuations \( V_i \) already referred to have an extension to

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the completion \( \overline{L} \) of \( L \) which we denote by \( \overline{V}_i \), and each such extension \( \overline{V}_i \) takes the value \( \epsilon \) on a minimal prime ideal \( \overline{P}_i \) of \( \overline{L} \). Let \( \delta_i \) denote the length of the primary component of \( (0) \) in \( \overline{L} \) with associated prime \( \overline{P}_i \). Then if \( x \in L \),

\[
e(xL) = \ell(L/xL) = \sum_{i=1}^{q} \delta_i [K_{V_i} : K] V_i(x)\]

where \( e(\cdot) \) is the multiplicity.

Now we turn to multiplicities and degree functions. Following Teissier, we will use mixed multiplicities. Let \( \alpha_1, \ldots, \alpha_d \) be \( \mathfrak{m} \)-primary ideals of \( Q \), and let \( M \) be a finitely generated \( Q \)-module. Then we define \( e(\alpha_1, \ldots, \alpha_d; M) \) as \( e(x_1, \ldots, x_d; M) \) where \( x_1, \ldots, x_d \) are independent general elements of \( \alpha_1, \ldots, \alpha_d \). Then we have the result that if \( L \) is as described earlier,

\[
e(\alpha_1, \ldots, \alpha_d) = e(x_dL) = e(\mathfrak{a}_dL),\]

the latter following since \( x_dL \) is a reduction of \( \mathfrak{a}_dL \). Further this latter remark also implies that, if \( V_i, v_i \) have the meanings given earlier, then \( V_i(x_d) = v_i(\alpha_d) \) where the latter denotes the minimum value of \( v_i(x) \) on \( \alpha_d \). We further note that \( e(\alpha_1, \ldots, \alpha_d; M) \) is a symmetric function of \( \alpha_1, \ldots, \alpha_d \) and, if \( \alpha_d' \) is another \( \mathfrak{m} \)-primary ideal of \( Q \), then

\[
e(\alpha_1, \ldots, \alpha_d, \alpha_d'; M) = e(\alpha_1, \ldots, \alpha_d; M) + e(\alpha_1, \ldots, \alpha_d; M)\]

we can now write down a formula for the multiplicity symbol

\[
e(\alpha_1, \ldots, \alpha_d; Q) = \sum_{i=1}^{q} \delta_i [K_{V_i} : K] v_i(\mathfrak{a}_d)\]

and similar formulae arising from the symmetry of the symbol. However this formula attains its full force if we introduce
degree functions. We define the degree function \( d(\alpha_1, \ldots, \alpha_d; x) \)
where \( x \) is an element of \( Q \) to be \( e(\alpha_1', \ldots, \alpha_d'; Q') \) where \( Q' = Q/x \)
and \( \alpha_i' = (\alpha_i + xQ)/xQ \). If \( Q \) is a domain, this can also be
written as \( e(x_1', \ldots, x_d'; x; Q) \) and we obtain the expression

\[
d(\alpha_1, \ldots, \alpha_d; x) = \sum_{i=1}^{q} \delta_i [K_{Y_i}:K] v_i(x).
\]

**APPENDIX**

First we prove a lemma which is well known.

**LEMMA.** Let \( B \) be a noether domain, \( y, z \) elements of \( B \) such that
\((y, z)\) has height 2. Let \( B' \) be the ring \( B[y/z] \) and let \( \mathfrak{P} \) be
the kernel of the map \( B[Y] \to B' \) in which \( Y = y/z \). Then \( \mathfrak{P} \)
contains \( w = zY - y \), and

\[
wB[Y] : (z^m, y^m) = \mathfrak{P}
\]

if \( m \) is sufficiently large. Further, if \( \mathfrak{m} \) is any prime ideal
of \( B \) containing \((y, z)\), then \( B'/\mathfrak{m}B' \cong (B/\mathfrak{m})[X] \), where \( X \) is an
indeterminate over \( B'/\mathfrak{m}B' \).

**Proof.** Let \( f(Y) \) be a polynomial of degree \( r \) over \( B \) such that
\( f(y/z) = 0 \). Then we can write \( f(Y) = F(Y, 1) \) where \( F(Y, Z) \) is
a homogeneous polynomial over \( B \) of degree \( r \) such that \( F(y, z) = 0 \).
Then

\[
z^r F(Y, Z) = F(zY, zZ) = F(yZ+(zY-yZ), zZ)
\]

\[
= F(yZ, zZ) + (zY-yZ)G(Y, Z) \quad \text{by Taylor's Theorem}
\]

\[
= z^r F(y, z) + (zY-yZ)G(Y, Z)
\]

whence, by putting \( Z = 1 \), we see that \( z^r f(Y) \in wB[Y] \). Also,

\[
y^r f(Y) = (y^r - z^r Y^r) f(Y) + Y^r z^r f(Y) \in wB[Y].
\]
But as the ascending sequence of ideals $wB[Y]:(y^r, z^r)$ becomes stationary for large $r$, it follows that

$$\mathfrak{p} = wB[Y]:(y^m, z^m) \quad m \text{ large.}$$

Hence $\mathfrak{p}$ is the radical of $wB[Y]$ and since $y, z \in \mathfrak{m}/\mathfrak{w}$, $w \in \mathfrak{m}B[Y]$, i.e. $\mathfrak{p} \subset wB[Y]$, which proves the result.

We now come to the main result of this appendix.

THEOREM. Let $(Q, \mathfrak{m}, k)$ be a local domain of dimension $d \geq 2$, and let $y_i, z_i$ $(i = 1, \ldots, d-1)$ be elements of $\mathfrak{m}$ such that $(y_i, z_1, \ldots, z_{d-1})$ is $\mathfrak{m}$-primary for $i = 1, \ldots, d-1$. Let $u_i = y_i/z_i$ and $B = Q[u_1, \ldots, u_{d-1}]$. Then

$$B/\mathfrak{m}B \cong k[x_1, \ldots, x_{d-1}]$$

where $x_1, \ldots, x_{d-1}$ are indeterminates over $k$, implying that $\mathfrak{m}B$ is prime.

Further let $L = B_{\mathfrak{m}B}$ and let $Q_{d-1}$ denote $Q[X_1, \ldots, X_{d-1}]$ localised at $\mathfrak{m}[X_1, \ldots, X_{d-1}]$. Let $\mathfrak{q}$ denote the kernel of the homomorphism $Q_{d-1} \rightarrow L$ in which $x_i \rightarrow u_i$. Let $w_i = z_i x_i - y_i$ and let $\mathfrak{x}$ be the ideal $(w_1, \ldots, w_{d-1})$. Then for $r$ large,

$$\mathfrak{m}^r \mathfrak{q} \subset \mathfrak{x}.$$

Proof. The proof will be by induction on $d$, the case $d=2$ following from the lemma. Now suppose that $d > 2$. Write $Q'$ for $Q[u_{d-1}]$ localised at $\mathfrak{m}[u_{d-1}]$, which is prime by the lemma. We first prove that $(y_i, z_1, \ldots, z_{d-2})Q'$ is $\mathfrak{m}Q'$-primary for $i = 1, \ldots, d-2$. Now, by the lemma, $Q' \cong Q(X_{d-1})/\mathfrak{q}'$, where $Q(X_{d-1})$ denotes $Q[X_{d-1}]$ localised at $\mathfrak{m}[X_{d-1}]$, and $\mathfrak{q}'$ is the radical of $w_{d-1}Q(X_{d-1})$. Hence it will be sufficient to show that $(w_{d-1}, y_i, z_1, \ldots, z_{d-2})$ is $\mathfrak{m}Q(X_{d-1})$-primary. Write
\[ C_i = y_1 Q(X_{d-1}) + z_1 Q(X_{d-1}) + \ldots + z_{d-2} Q(X_{d-1}). \]

Then the minimal prime ideals of \( C_i \) are generated by elements of \( Q \) and so can only contain \( w_{d-1} \) if it contains \( y_{d-1}, z_{d-1} \). Since \( C_i + z_{d-1} Q(X_{d-1}) \) is \( m \)-primary, \( \dim C_i = 1 \), and since \( w_{d-1} \) belongs to no minimal prime of \( C_i \), the result now follows.

Now we consider the first statement of the theorem. It is clearly equivalent to the statement that if \( f(x_1, \ldots, x_{d-1}) \) is a polynomial over \( Q \) such that \( f(u_1, \ldots, u_{d-1}) = 0 \), then all the coefficients of \( f \) belong to \( m \). Suppose there is a coefficient of \( f \) not in \( m \). Then if we consider the polynomial \( f(x_1, \ldots, x_{d-2}, u_{d-1}) \) as a polynomial with coefficients in \( Q' \), then the lemma implies that this has a coefficient not in \( m \cdot Q' \).

But \( Q' \) has dimension \( d-1 \) and the conditions of the theorem apply. Hence by our inductive hypothesis \( f(u_1, \ldots, u_{d-1}) \neq 0 \).

We are now in a position to construct \( L \). Consider the homomorphism \( Q_{d-1} \to L \). This can be factored as the product of the homomorphism \( Q_{d-1} \to Q'_{d-2} \) in which \( X_{d-1} \to u_{d-1} \) and the homomorphism \( Q'_{d-2} \to L \). Denote by \( 0 \) the kernel of the homomorphism \( Q_{d-1} \to Q'_{d-2} \). Applying the inductive hypothesis to the second factor, we see that, for \( r \) large,

\[ m^R \mathcal{P} \subset 0 + (w_1, \ldots, w_{d-2}) \]

while, by the lemma,

\[ (y_{d-1}^m, z_{d-1}^m) 0 \subset w_{d-1} Q_{d-1}. \]

Hence

\[ (y_{d-1}^m, z_{d-1}^m) m^R \mathcal{P} \subset (w_1, \ldots, w_{d-1}) = \mathcal{F}. \]

But by reordering the suffixes \( 1, \ldots, d-1 \), we can replace \( d-1 \) on the left hand side by \( i \) \( (i = 1, \ldots, d-2) \). Hence if \( m, r \) are
large enough,

$$(y_1^m, \ldots, y_{d-1}^m, z_{d-2}^m, \ldots, z_{d-1}^m)^m \mathcal{P} \subset \mathcal{I}$$

and the result follows since the first factor is $m$-primary.