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Kyoto University
ON THE CANONICAL MODULES

愛媛大 理 青山陽一 ( Yôichi Aoyama )

A ring will always mean a commutative noetherian ring with unit. Let $R$ be a ring, $M$ a finitely generated $R$-module and $N$ a submodule of $M$. We denote by $\text{Min}_R(M)$ the set of minimal elements in $\text{Supp}_R(M)$ and put $U_M(N) = \bigcap Q$ where $Q$ runs through all the primary components of $N$ in $M$ such that $\dim M/Q = \dim M/N$. Let $T$ be an $R$-module and $\mathfrak{a}$ an ideal of $R$. $E_R(T)$ denotes an injective envelope of $T$ and $H^1_\mathfrak{a}(T)$ is the 1-th local cohomology module of $T$ with respect to $\mathfrak{a}$. We denote by $\hat{\mathfrak{a}}$ the Jacobson radical adic completion over a semi-local ring. For a ring $R$, $Q(R)$ denotes the total quotient ring of $R$. Throughout this note $A$ denotes a local ring of dimension $d$ and with maximal ideal $\mathfrak{m}$.

Definition([7, Definition 5.6]). An $A$-module $K$ is called a canonical module of $A$ if $K \otimes_A \hat{\mathfrak{a}} \cong \text{Hom}_A(H^d_\mathfrak{m}(A), E_A(A/\mathfrak{m}))$.

For elementary properties of canonical modules, we refer the reader to [6, §6], [7, 5 Vortrag und 6 Vortrag] and [2, §1]. It is not obvious that the localization of a canonical module is a canonical module of the localization ring, which was known only
for local rings with dualizing complexes, and Ogoma [9] showed that there is a non-acceptable (hence without dualizing complex) local ring with canonical module. Our purposes are to prove that $K_p$ is a canonical module of $A_p$ for every $p$ in $\text{Supp}_A(K)$ ($A$ is a local ring with canonical module $K$) and to consider endomorphism rings of canonical modules.

**Lemma 1** (Corollary to [5, Theorem 1]). Let $B$ be a faithfully flat local $A$-algebra with maximal ideal $m$. Then:

1. If $B/mB$ is an artinian Gorenstein ring, then $E_A(A/m) \otimes_A B = E_B(B/m)$.
2. If $T$ is an $A$-module such that $T \otimes_A B = E_B(B/m)$, then $T = E_A(A/m)$ and $B/mB$ is an artinian Gorenstein ring.

**Theorem 2** ([4]). Assume that $A$ has a canonical module $K$ and let $B$ be a faithfully flat local $A$-algebra. Then the following are equivalent:

(a) $B/mB$ is a Gorenstein ring.
(b) $K \otimes_A B$ is a canonical module of $B$ and $B/mB$ is a Cohen-Macaulay ring.

(Proof) Suppose that $B/mB$ is a Cohen-Macaulay ring and let $y_1, \ldots, y_r$ be a system of elements in $m$, the maximal ideal of $B$, which is a maximal $B/mB$-regular sequence ($r = \dim B/mB$).

Let $R = A[X_1, \ldots, X_r]$ with indeterminates $X_1, \ldots, X_r$ over $A$ and let $f$ be the natural $A$-algebra homomorphism from $R$ to $B$ such that $f(X_i) = y_i$ for $i = 1, \ldots, r$. Then $f$ is a flat local homomorphism. By [7, Korollar 5.12], $C = K \otimes_A R$ is a canonical module of $R$. Hence we may assume that $B/mB$
is artinian. Furthermore we may assume that $A$ and $B$ are both complete. In this case it is shown that $K \otimes_A B$ is a canonical module of $B$ if and only if $E_A(A/m) \otimes_A B \cong E_B(B/n)$ ([2, Proof of Proposition 4.1]). Hence the assertion follows from Lemma 1. (Q.E.D.)

Suppose that $A$ has a canonical module $K$. Let $M$ be a finitely generated $A$-module and $h_M$ the natural map from $M$ to $\text{Hom}_A(\text{Hom}_A(M,K),K)$.

Proposition 3([2, (1.11)]). The following are equivalent:
(a) The map $h_M$ is an isomorphism.
(b) $\hat{M}$ is $(S_2)$ and $\dim A/\mathfrak{p} = d$ for every $\mathfrak{p}$ in $\text{Min}_A(M)$.

Corollary 4([1, Proposition 2]). $A \cong \text{Hom}_A(K,K)$ if and only if $\hat{A}$ is $(S_2)$.

Next we show some elementary properties of the endomorphism ring of a canonical module. Assume that $A$ has a canonical module $K$ and put $H = \text{End}_A(K)$.

Theorem 5([2, Theorem 3.2]). The following statements hold for $H$:
(1) $H$ is a semi-local ring which is a finitely generated $A$-module and $A/U \subseteq H \subseteq Q(A/U)$ where $U = U_A(0) = \text{ann}_A(K)$.
(2) Every maximal chain of prime ideals in $H$ is of length $d$.
(3) $\hat{H}$ is $(S_2)$.
(4) For every maximal ideal $n$ of $H$, $K_n$ is a canonical module of $H_n$. (K is an $H$-module by the usual way.)
(5) $\dim_A \text{Coker}(A+H) \leq d - 2$. 

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(Proof) We may assume that \( \text{ann}_A(K) = U_A(0) = 0 \).

(1) Let \( p \) be a prime ideal of \( A \) with \( \dim A/p = d \) and \( q \) a minimal prime ideal of \( p\hat{A} \). Then \( \dim \hat{A}/q = d \) and \( \hat{K}_q \) is a canonical module of \( \hat{A}/q \). Since \( \dim \hat{A}/q = 0 \), \( \hat{K}_q \cong E_A(\hat{A}/q) \).

Since \( K \otimes_{A_p} \hat{A}_p \cong \hat{K}_q \), \( K = E_A(A/p) \) by Lemma 1(2). Let \( \text{Ass}(A) = \{p_1, \ldots, p_t\} \) and \( S = A \setminus \bigcup_{i=1}^t p_i \), the set of non-zerodivisors of \( A \). Since \( K \) is torsion free, so is \( H \) and the natural map \( H \to S^{-1}H \) is injective. Since \( S^{-1}K \cong \bigoplus_{i=1}^t K_{p_i} \cong \bigoplus_{i=1}^t E_A(A/p_i) \), \( S^{-1}H \cong \text{Hom}_A(S^{-1}K, S^{-1}K) \cong \bigoplus_{i=1}^t A_{p_i} \cong Q(A) \).

(2) Because \( A \) is unmixed.

(3) Because \( \hat{K} \) is \( (S_2) \).

(4) The map \( h_K : K \to \text{Hom}_A(H, K) \) is an isomorphism by Proposition 3. Hence the assertion follows from [7, Satz 5.12] and (3).

(5) We may assume that \( A \) is complete. Let \( p \) be a prime ideal such that \( \height p \leq 1 \). Then \( A_p \) is Cohen-Macaulay and \( K_p \) is a canonical module of \( A_p \) because \( A \) is complete and \( U_A(0) = 0 \). Hence \( A_p = H_p \), that is, \( \text{Coker}(A\to H)_p = 0 \), which means \( \dim_A \text{Coker}(A\to H) \leq d - 2 \). (Q.E.D.)

Theorem 6([2, Theorem 4.2]). Let \( (A, m) \to (B, n) \) be a flat local homomorphism and \( M \) an \( A \)-module. If \( M \otimes_A B \) is a canonical module of \( B \), then \( M \) is a canonical module of \( A \).

Corollary 7([2, Corollary 4.3]). Assume that \( A \) has a canonical module \( K \) and let \( p \) be an element of \( \text{Supp}_A(K) \). Then \( K_p \) is a canonical module of \( A_p \) and \( \hat{A}/q \otimes_{\hat{A}} q \) is a Gorenstein ring for every minimal prime ideal \( q \) of \( p\hat{A} \).

Before proving Theorem 6, we show two lemmas.
Lemma 8. Assume that $A$ is complete. Let $T$ be a finitely generated $(S_2)$ $A$-module such that $\dim A/\mathfrak{p} = d$ for every $\mathfrak{p}$ in $\text{Min}_A(T)$ and $H^d_{\mathfrak{m}}(T) \cong E_A(A/\mathfrak{m})$. Then $T$ is a canonical module of $A$. In this case $A$ is $(S_2)$.

(Proof) By Proposition 3, the map $h_T$ is an isomorphism. Since $\text{Hom}_A(T,K) \cong \text{Hom}_A(H^d_{\mathfrak{m}}(T),E_A(A/\mathfrak{m})) \cong \text{Hom}_A(E_A(A/\mathfrak{m}),E_A(A/\mathfrak{m})) \cong A$, $T \cong \text{Hom}_A(A,K) \cong K$, a canonical module of $A$. (Q.E.D.)

Lemma 9. Let $R$ be a finite over-ring of $A$ such that $\dim_A R/A \leq d - 2$ and $\dim R/\mathfrak{p} = d$ for every maximal ideal $\mathfrak{p}$ of $R$. If $T$ is a finitely generated $R$-module such that $T_\mathfrak{p}$ is a canonical module of $R_\mathfrak{p}$ for every maximal ideal $\mathfrak{p}$ of $R$, then $T$, as an $A$-module, is a canonical module of $A$.

(Proof) We may assume that $A$ is complete. For every maximal ideal $\mathfrak{p}$ of $R$, $\text{Hom}_A(R,K)_\mathfrak{p}$ is a canonical module of $R_\mathfrak{p}$ by [7, Satz 5.12] ($K$ is a canonical module of $A$). Hence $T_\mathfrak{p} \cong \text{Hom}_A(R,K)_\mathfrak{p}$ for every maximal ideal $\mathfrak{p}$ of $R$ and therefore $T \cong \text{Hom}_A(R,K)$. Since $\dim_A R/A \leq d - 2$, we have $\text{Hom}_A(R/A,K) = 0$ and $\text{Ext}^1_A(R/A,K) = 0$ (cf. [2, (1.10)]). Hence, from the exact sequence $0 \to A \to R \to R/A \to 0$, we have $\text{Hom}_A(R,K) \cong \text{Hom}_A(A,K) \cong K$, a canonical module of $A$. (Q.E.D.)

(Proof of Theorem 6) We may assume that $A$ and $B$ are both complete and $mB$ is $n$-primary. Let $K$ (resp. $L$) be a canonical module of $A$ (resp. $B$).

(I) The case that $B$ is $(S_2)$: Since $B$ is $(S_2)$, $B \cong \text{Hom}_B(L,L)$, i.e., $H^n_B(L) \cong E_B(B/n)$. Since $H^n_{\mathfrak{m}}(M) \cong H^n_{\mathfrak{m}}(M \otimes_A B) \cong H^n_B(L) \cong E_B(B/n)$, $H^n_{\mathfrak{m}}(M) \cong E_A(A/\mathfrak{m})$ by Lemma 1(2). Since $L$ is $(S_2)$,
so is $M$. Since $\text{Ass}_B(L) = \{ q \in \text{Spec}(B) \mid \dim B/q = d \}$, $\text{Ass}_A(M) = \{ p \in \text{Spec}(A) \mid \dim A/p = d \}$. Hence we have $M \cong K$ by Lemma 8.

(II) The general case: Since $\text{Ass}_A(M) = \{ p \in \text{Spec}(A) \mid \dim A/p = d \}$ and $M_p \cong E_A(A/p)$ for every $p$ in $\text{Ass}_A(M)$ (cf. Proof of Theorem 5(1)), we have $\text{ann}_A(M) = U_A(0)$. Hence we may assume that $U_A(0) = 0$ and $U_B(0) = 0$. Put $R = \text{End}_A(M)$ and $S = \text{End}_B(L)$. Since $R \otimes_A B \cong S$ is a finite over-ring of $B$, $R$ is a finite over-ring of $A$. For every maximal ideal $p$ of $R$, $\dim R_p = d$ because $A$ is unmixed. We have $\dim_A R/A \leq d - 2$ because $\dim_B S/B \leq d - 2$. Let $p$ be a maximal ideal of $R$ and $q$ a maximal ideal of $S$ lying over $p$. Since $M_p \otimes_R S_q \cong L_q$ is a canonical module of $S_q$ by Theorem 5(4) and $S_q$ is $(S_2)$ by Theorem 5(3), $M_p$ is a canonical module of $R_p$ by the case (I). Hence we have that $M$ is a canonical module of $A$ by Lemma 9. (Q.E.D.)

Remark. Goto (Nihon University) proved the following lemma and gave another proof of Theorem 6. ([3, Appendix])

**Lemma.** Let $(A,m) \rightarrow (B,n)$ be a flat local homomorphism such that $mB$ is $n$-primary. If there is a finitely generated $A$-module $T$ such that $T \otimes_A B$ is a canonical module of $B$, then $B/mB$ is a Gorenstein ring.

By virtue of Corollary 7, we can prove the following proposition by induction on $\dim A$ (cf. [1, Proof of Proposition 2]). Assume that $A$ has a canonical module $K$. For a finitely generated $A$-module $M$, $h_M$ denotes the natural map from $M$ to
Proposition 10([2, Proposition 4.4]). The following are equivalent:

(a) The map \( h_M \) is an isomorphism.
(b) \( \hat{M} \) is \((S_2)\) and \( \dim A/p = d \) for every \( p \) in \( \text{Min}_A(M) \).
(c) \( M \) is \((S_2)\) and \( \dim A/p = d \) for every \( p \) in \( \text{Min}_A(M) \).

Corollary 11([9, Proposition 4.2] and [4]). The following are equivalent:

(a) \( A \cong \text{Hom}_A(K,K) \).
(b) \( \hat{A} \) is \((S_2)\).
(c) \( A \) is \((S_2)\).

Remark. The implication (c) \( \Rightarrow \) (a) was first proved by Ogoma (Kochi University), not by induction. (See [9, §4]. cf. [3, (2)])

Corollary 12([4]). Assume that \( A \) has a canonical module and \( \dim A/p = d \) for every \( p \) in \( \text{Min}(A) \). Then the \((S_2)\)-locus of \( A \) is open in \( \text{Spec}(A) \).

Corollary 13([4]). Assume that \( A \) has a canonical module. Let \( (A,m) \rightarrow (B,n) \) be a flat local homomorphism such that \( B/mB \) is a Gorenstein ring.

1. Let \( M \) be a finitely generated \((S_2)\) \( A \)-module such that \( \dim A/p = d \) for every \( p \) in \( \text{Min}_A(M) \). Then \( M \otimes_A B \) is \((S_2)\) and \( \dim B/q = \dim B \) for every \( q \) in \( \text{Min}_B(M \otimes_A B) \).

2. If \( A \) is \((S_2)\), then \( B \) is also \((S_2)\).

Next we show that the endomorphism ring of a canonical module is characterized by the properties described in Theorem 5.
Theorem 14([4]). Assume that $A$ has a canonical module $K$.

Let $R$ be a ring satisfying the following conditions:

(i) $R$ is a finite $(S_2)$ over-ring of $A/U_A(0)$,
(ii) For every maximal ideal $\mathfrak{n}$ of $R$, $\dim R_{\mathfrak{n}} = d$, and
(iii) $\dim_A \operatorname{Coker}(A \to R) \leq d - 2$.

Then $R \cong \operatorname{End}_A(K)$ as $A$-algebras.

(Proof) We may assume that $U_A(0) = 0$. Put $L = \operatorname{Hom}_A(R, K)$.

Then $L_{\mathfrak{n}}$ is a canonical module of $R_{\mathfrak{n}}$ for every maximal ideal $\mathfrak{n}$ of $R$. By Lemma 9, we have $L \cong K$. From this isomorphism, we have an $A$-algebra isomorphism $\operatorname{End}_A(K) \cong \operatorname{End}_A(L)$. Since $\operatorname{End}_A(K)$ is commutative, so is $\operatorname{End}_A(L)$ and $\operatorname{End}_A(L) = \operatorname{End}_R(L)$.

Since $R$ is $(S_2)$, $R \cong \operatorname{End}_R(L)$. Hence we have $R \cong \operatorname{End}_A(K)$ as $A$-algebras. (Q.E.D.)

In the following we assume that $A$ has a canonical module $K$, $d \geq 2$ and $U_A(0) = 0$. Put $H = \operatorname{End}_A(K)$ and $c = A:A H$, the conductor. Let $T$ be the $c$-transform of $A$, i.e., $T = \{ x \in Q(A) | ct x \subseteq A \text{ for some } t \}$. Let $q$ be a prime ideal of $\hat{A}$ containing $c\hat{A}$ and $p$ an associated prime ideal of $\hat{A}_q$. Since $U_{\hat{A}}(0) = U_A(0)\hat{A} = 0$ and height $c \geq 2$, we have $\dim \hat{A}_q/p \geq 2$. Hence by [8, Proposition(2.7)] we have:

(15.1) $T$ is a finitely generated $A$-module.

The following two assertions are obvious:

(15.2) $\dim_A T/A \leq d - 2$.
(15.3) $T$ is $(S_2)$.

Hence, from Theorem 14, we obtain the following
Proposition 16([4]). \( T \cong H \) as \( A \)-algebras.

We denote by \( A^G \) the global transform of \( A \), i.e., \( A^G = \{ x \in \text{Q}(A) \mid m^t x \subseteq A \text{ for some } t \} \). Since \( U_A(0) = 0 \) and \( d \geq 2 \), \( A^G \) is a finitely generated \( A \)-module by [8, Proposition (2.3)].

Corollary 17([4]). \( A^G \cong H \) as \( A \)-algebras if and only if depth \( A_p \geq \min \{ 2, \dim A_p \} \) for every non-maximal prime ideal \( p \) of \( A \). In particular, if \( H^i_m(A) \) is of finite length for \( i \neq d \), \( A^G \cong H \) as \( A \)-algebras.

Ehime University

References


