ON THE CANONICAL MODULES

A ring will always mean a commutative noetherian ring with unit. Let $R$ be a ring, $M$ a finitely generated $R$-module and $N$ a submodule of $M$. We denote by $\text{Min}_R(M)$ the set of minimal elements in $\text{Supp}_R(M)$ and put $U_M(N) = \bigcap Q$ where $Q$ runs through all the primary components of $N$ in $M$ such that $\dim M/Q = \dim M/N$. Let $T$ be an $R$-module and $\mathfrak{a}$ an ideal of $R$. $E_R(T)$ denotes an injective envelope of $T$ and $\mathcal{H}^1_\mathfrak{a}(T)$ is the $1$-th local cohomology module of $T$ with respect to $\mathfrak{a}$. We denote by $\mathfrak{a}$ the Jacobson radical adic completion over a semi-local ring. For a ring $R$, $Q(R)$ denotes the total quotient ring of $R$. Throughout this note $A$ denotes a local ring of dimension $d$ and with maximal ideal $\mathfrak{m}$.

Definition([7, Definition 5.6]). An $A$-module $K$ is called a canonical module of $A$ if $K \otimes_A \mathcal{A} \cong \text{Hom}_A(\mathcal{H}^d_\mathfrak{m}(A), E_A(A/\mathfrak{m}))$.

For elementary properties of canonical modules, we refer the reader to [6, §6], [7, 5 Vortrag und 6 Vortrag] and [2, §1]. It is not obvious that the localization of a canonical module is a canonical module of the localization ring, which was known only
for local rings with dualizing complexes, and Ogoma [9] showed that there is a non-acceptable (hence without dualizing complex) local ring with canonical module. Our purposes are to prove that $K_p$ is a canonical module of $A_p$ for every $p$ in $\text{Supp}_A(K)$ ($A$ is a local ring with canonical module $K$) and to consider endomorphism rings of canonical modules.

**Lemma 1** (Corollary to [5, Theorem 1]). Let $B$ be a faithfully flat local $A$-algebra with maximal ideal $n$. Then:

1. If $B/mB$ is an artinian Gorenstein ring, then $E_A(A/m) \otimes_A B \cong E_B(B/n)$.

2. If $T$ is an $A$-module such that $T \otimes_A B \cong E_B(B/n)$, then $T \cong E_A(A/m)$ and $B/mB$ is an artinian Gorenstein ring.

**Theorem 2** ([4]). Assume that $A$ has a canonical module $K$ and let $B$ be a faithfully flat local $A$-algebra. Then the following are equivalent:

(a) $B/mB$ is a Gorenstein ring.

(b) $K \otimes_A B$ is a canonical module of $B$ and $B/mB$ is a Cohen-Macaulay ring.

(Proof) Suppose that $B/mB$ is a Cohen-Macaulay ring and let $y_1, \ldots, y_r$ be a system of elements in $n$, the maximal ideal of $B$, which is a maximal $B/mB$-regular sequence ($r = \dim B/mB$). Let $R = A[\overline{x}_1, \ldots, \overline{x}_r]_{(m, \overline{x}_1, \ldots, \overline{x}_r)}$ with indeterminates $x_1, \ldots, x_r$ over $A$ and let $f$ be the natural $A$-algebra homomorphism from $R$ to $B$ such that $f(x_i) = y_i$ for $i = 1, \ldots, r$. Then $f$ is a flat local homomorphism. By [7, Korollar 5.12], $C = K \otimes_A R$ is a canonical module of $R$. Hence we may assume that $B/mB$
is artinian. Furthermore we may assume that $A$ and $B$ are both complete. In this case it is shown that $K\Theta_A B$ is a canonical module of $B$ if and only if $E_A(A/\mathfrak{m})\Theta_A B \cong E_B(B/\mathfrak{n})$ ([2, Proof of Proposition 4.1]). Hence the assertion follows from Lemma 1. (Q.E.D.)

Suppose that $A$ has a canonical module $K$. Let $M$ be a finitely generated $A$-module and $h_M$ the natural map from $M$ to $\text{Hom}_A(\text{Hom}_A(M,K),K)$.

**Proposition 3**([2, (1.11)]). The following are equivalent:

(a) The map $h_M$ is an isomorphism.

(b) $\hat{M}$ is $(S_2)$ and $\dim A/\mathfrak{p} = d$ for every $\mathfrak{p}$ in $\text{Min}_A(M)$.

**Corollary 4**([1, Proposition 2]). $A \cong \text{Hom}_A(K,K)$ if and only if $\hat{A}$ is $(S_2)$.

Next we show some elementary properties of the endomorphism ring of a canonical module. Assume that $A$ has a canonical module $K$ and put $H = \text{End}_A(K)$.

**Theorem 5**([2, Theorem 3.2]). The following statements hold for $H$:

1. $H$ is a semi-local ring which is a finitely generated $A$-module and $A/\mathfrak{U} \subseteq H \subseteq Q(A/\mathfrak{U})$ where $\mathfrak{U} = U_A(0) = \text{ann}_A(K)$.
2. Every maximal chain of prime ideals in $H$ is of length $d$.
3. $\hat{H}$ is $(S_2)$.
4. For every maximal ideal $\mathfrak{n}$ of $H$, $K_{\mathfrak{n}}$ is a canonical module of $H_{\mathfrak{n}}$. (K is an $H$-module by the usual way.)
5. $\dim_A \text{Coker}(A+H) \leq d - 2$. 


(Proof) We may assume that \( \text{ann}_A(K) = U_A(0) = 0 \).

(1) Let \( p \) be a prime ideal of \( A \) with \( \dim A/p = d \) and \( q \) a minimal prime ideal of \( p\hat{A} \). Then \( \dim \hat{A}/q = d \) and \( \hat{R}_q \) is a canonical module of \( \hat{A}/q \). Since \( \dim \hat{A}/q = 0 \), \( \hat{R}_q \cong E_A(\hat{A}/q) \).

Since \( \bigoplus_{p \in \mathcal{P}} \hat{A}/p = \hat{R} \), \( K_p \cong E_A(A/p) \) by Lemma 1(2). Let \( \text{Ass}(A) = \{p_1, \ldots, p_t\} \) and \( S = A \setminus \bigcup_{i=1}^t p_i \), the set of non-zerodivisors of \( A \). Since \( K \) is torsion free, so is \( H \) and the natural map \( H \to S^{-1}H \) is injective. Since \( S^{-1}K \cong \bigoplus_{i=1}^t E_A(A/p_i) \), \( S^{-1}H \cong \text{Hom}_A(S^{-1}K, S^{-1}K) \cong \bigoplus_{i=1}^t E_A(A/p_i) \cong Q(A) \).

(2) Because \( A \) is unmixed.

(3) Because \( \hat{K} \) is \( (S_2^c) \).

(4) The map \( h_K : K \to \text{Hom}_A(H, K) \) is an isomorphism by Proposition 3. Hence the assertion follows from [7, Satz 5.12] and (3).

(5) We may assume that \( A \) is complete. Let \( p \) be a prime ideal such that \( \text{height } p \leq 1 \). Then \( A_p \) is Cohen-Macaulay and \( K_p \) is a canonical module of \( A_p \) because \( A \) is complete and \( U_A(0) = 0 \). Hence \( A_p = H_p \), that is, \( \text{Coker}(A-H)_p = 0 \), which means \( \dim A \text{ Coker}(A-H) \leq d-2 \). (Q.E.D.)

Theorem 6([2, Theorem 4.2]). Let \( (A, \mathfrak{m}) \to (B, \mathfrak{n}) \) be a flat local homomorphism and \( M \) an \( A \)-module. If \( M \otimes_A B \) is a canonical module of \( B \), then \( M \) is a canonical module of \( A \).

Corollary 7([2, Corollary 4.3]). Assume that \( A \) has a canonical module \( K \) and let \( p \) be an element of \( \text{Supp}_A(K) \). Then \( K_p \) is a canonical module of \( A_p \) and \( \hat{A}/p\hat{A}_q \) is a Gorenstein ring for every minimal prime ideal \( q \) of \( p\hat{A} \).

Before proving Theorem 6, we show two lemmas.
Lemma 8. Assume that $A$ is complete. Let $T$ be a finitely generated $(S_2)$ $A$-module such that $\dim A/p = d$ for every $p$ in $\text{Min}_A(T)$ and $H^d_m(T) \cong E_A(A/m)$. Then $T$ is a canonical module of $A$. In this case $A$ is $(S_2)$.

(Proof) By Proposition 3, the map $h_T$ is an isomorphism. Since $\text{Hom}_A(T,K) \cong \text{Hom}_A(H^d_m(T),E_A(A/m)) \cong \text{Hom}_A(E_A(A/m),E_A(A/m)) \cong A$, $T \cong \text{Hom}_A(A,K) \cong K$, a canonical module of $A$. (Q.E.D.)

Lemma 9. Let $R$ be a finite over-ring of $A$ such that $\dim_A R/A \leq d - 2$ and $\dim R_p = d$ for every maximal ideal $p$ of $R$. If $T$ is a finitely generated $R$-module such that $T_p$ is a canonical module of $R_p$ for every maximal ideal $p$ of $R$, then $T$, as an $A$-module, is a canonical module of $A$.

(Proof) We may assume that $A$ is complete. For every maximal ideal $p$ of $R$, $\text{Hom}_A(R,K)_p$ is a canonical module of $R_p$ by [7, Satz 5.12] (K is a canonical module of $A$). Hence $T_p \cong \text{Hom}_A(R,K)_p$ for every maximal ideal $p$ of $R$ and therefore $T \cong \text{Hom}_A(R,K)$. Since $\dim_A R/A \leq d - 2$, we have $\text{Hom}_A(R/A,K) = 0$ and $\text{Ext}^1_A(R/A,K) = 0$ (cf. [2, (1.10)]). Hence, from the exact sequence $0 \to A \to R \to R/A \to 0$, we have $\text{Hom}_A(R,K) \cong \text{Hom}_A(A,K) \cong K$, a canonical module of $A$. (Q.E.D.)

(Proof of Theorem 6) We may assume that $A$ and $B$ are both complete and $mB$ is $n$-primary. Let $K$ (resp. $L$) be a canonical module of $A$ (resp. $B$).

(I) The case that $B$ is $(S_2)$: Since $B$ is $(S_2)$, $B \cong \text{Hom}_B(L,L)$, i.e., $H^d_n(L) \cong E_B(B/n)$. Since $H^d_m(M) \otimes_A B \cong H^d_n(M \otimes_A B) \cong H^d_n(L) \cong E_B(B/n)$, $H^d_m(M) \cong E_A(A/m)$ by Lemma 1(2). Since $L$ is $(S_2)$,
so is $M$. Since $\text{Ass}_B(L) = \{ q \in \text{Spec}(B) \mid \dim B/q = d \}$, $\text{Ass}_A(M) = \{ p \in \text{Spec}(A) \mid \dim A/p = d \}$. Hence we have $M \cong K$ by Lemma 8.

(II) The general case: Since $\text{Ass}_A(M) = \{ p \in \text{Spec}(A) \mid \dim A/p = d \}$ and $M_p \cong E_A(A/p)$ for every $p$ in $\text{Ass}_A(M)$ (cf. Proof of Theorem 5(1)), we have $\text{ann}_A(M) = U_A(0)$. Hence we may assume that $U_A(0) = 0$ and $U_B(0) = 0$. Put $R = \text{End}_A(M)$ and $S = \text{End}_B(L)$. Since $R \otimes_A B \cong S$ is a finite over-ring of $B$, $R$ is a finite over-ring of $A$. For every maximal ideal $p$ of $R$, $\dim R_p = d$ because $A$ is unmixed. We have $\dim_A R/A \leq d - 2$ because $\dim_B S/B \leq d - 2$. Let $p$ be a maximal ideal of $R$ and $q$ a maximal ideal of $S$ lying over $p$. Since $M_p \otimes R_p S_q \cong L_q$ is a canonical module of $S_q$ by Theorem 5(4) and $S_q$ is $(S_2)$ by Theorem 5(3), $M_p$ is a canonical module of $R_p$ by the case (I). Hence we have that $M$ is a canonical module of $A$ by Lemma 9. (Q.E.D.)

Remark. Goto (Nihon University) proved the following lemma and gave another proof of Theorem 6. ([3, Appendix])

Lemma. Let $(A, m) \rightarrow (B, n)$ be a flat local homomorphism such that $mB$ is $n$-primary. If there is a finitely generated $A$-module $T$ such that $T \otimes_A B$ is a canonical module of $B$, then $B/mB$ is a Gorenstein ring.

By virtue of Corollary 7, we can prove the following proposition by induction on $\dim A$ (cf. [1, Proof of Proposition 2]). Assume that $A$ has a canonical module $K$. For a finitely generated $A$-module $M$, $h_M$ denotes the natural map from $M$ to
\[ \text{Hom}_A(\text{Hom}_A(M,K),K). \]

**Proposition 10** ([2, Proposition 4.4]). The following are equivalent:

(a) The map \( h_M \) is an isomorphism.

(b) \( \hat{M} \) is \( (S_2) \) and \( \dim A/p = d \) for every \( p \) in \( \text{Min}_A(M) \).

(c) \( M \) is \( (S_2) \) and \( \dim A/p = d \) for every \( p \) in \( \text{Min}_A(M) \).

**Corollary 11** ([9, Proposition 4.2] and [4]). The following are equivalent:

(a) \( A \cong \text{Hom}_A(K,K) \).

(b) \( \hat{A} \) is \( (S_2) \).

(c) \( A \) is \( (S_2) \).

**Remark.** The implication (c) \( \Rightarrow \) (a) was first proved by Ogoma (Kochi University), not by induction. (See [9, §4]. cf. [3, (\( \approx \))] )

**Corollary 12** ([4]). Assume that \( A \) has a canonical module and \( \dim A/p = d \) for every \( p \) in \( \text{Min}(A) \). Then the \( (S_2) \)-locus of \( A \) is open in \( \text{Spec}(A) \).

**Corollary 13** ([4]). Assume that \( A \) has a canonical module. Let \( (A,m) \to (B,n) \) be a flat local homomorphism such that \( B/mB \) is a Gorenstein ring.

1. Let \( M \) be a finitely generated \( (S_2) \) \( A \)-module such that \( \dim A/p = d \) for every \( p \) in \( \text{Min}_A(M) \). Then \( M \otimes_A B \) is \( (S_2) \) and \( \dim B/q = \dim B \) for every \( q \) in \( \text{Min}_B(M \otimes_A B) \).

2. If \( A \) is \( (S_2) \), then \( B \) is also \( (S_2) \).

Next we show that the endomorphism ring of a canonical module is characterized by the properties described in Theorem 5.
Theorem 14([4]). Assume that $A$ has a canonical module $K$.

Let $R$ be a ring satisfying the following conditions:

(i) $R$ is a finite $(S_2)$ over-ring of $A/U_A(0)$,

(ii) For every maximal ideal $\mathfrak{n}$ of $R$, $\dim R_{\mathfrak{n}} = d$, and

(iii) $\dim_A \text{Coker}(A \to R) \leq d - 2$.

Then $R \cong \text{End}_A(K)$ as $A$-algebras.

(Proof) We may assume that $U_A(0) = 0$. Put $L = \text{Hom}_A(R,K)$.
Then $L_{\mathfrak{n}}$ is a canonical module of $R_{\mathfrak{n}}$ for every maximal ideal $\mathfrak{n}$ of $R$. By Lemma 9, we have $L \cong K$. From this isomorphism, we have an $A$-algebra isomorphism $\text{End}_A(K) \cong \text{End}_A(L)$. Since $\text{End}_A(K)$ is commutative, so is $\text{End}_A(L)$ and $\text{End}_A(L) = \text{End}_R(L)$. Since $R$ is $(S_2)$, $R \cong \text{End}_R(L)$. Hence we have $R \cong \text{End}_A(K)$ as $A$-algebras. (Q.E.D.)

In the following we assume that $A$ has a canonical module $K$, $d \geq 2$ and $U_A(0) = 0$. Put $H = \text{End}_A(K)$ and $c = A : A H$, the conductor. Let $T$ be the $c$-transform of $A$, i.e., $T = \{ x \in Q(A) \mid c^tx \subseteq A \text{ for some } t \}$. Let $q$ be a prime ideal of $\hat{A}$ containing $c\hat{A}$ and $\mathfrak{p}$ an associated prime ideal of $\hat{A}_q$. Since $U_A(0) = U_A(0)\hat{A} = 0$ and height $c \geq 2$, we have $\dim \hat{A}_{q/\mathfrak{p}} \geq 2$. Hence by [8, Proposition(2.7)] we have:

(15.1) $T$ is a finitely generated $A$-module.

The following two assertions are obvious:

(15.2) $\dim_A T/A \leq d - 2$.

(15.3) $T$ is $(S_2)$.

Hence, from Theorem 14, we obtain the following
Proposition 16([4]). $T \cong H$ as $A$-algebras.

We denote by $A^G$ the global transform of $A$, i.e., $A^G = \{ x \in Q(A) \mid m^t x \subseteq A \text{ for some } t \}$. Since $U^A_A(0) = 0$ and $d \geq 2$, $A^G$ is a finitely generated $A$-module by [8, Proposition (2.3)].

Corollary 17([4]). $A^G \cong H$ as $A$-algebras if and only if depth $A_p \geq \min \{ 2, \dim A_p \}$ for every non-maximal prime ideal $p$ of $A$. In particular, if $H^i_m(A)$ is of finite length for $i \neq d$, $A^G \cong H$ as $A$-algebras.

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References


