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Kyoto University
Pointwise Completeness

and

Controllability by Linear Delay Feedback

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1. Introduction

Let a control system be described by

$$\frac{d}{dt} x(t) = Ax(t) + u(t)$$

on a Banach space $X$. We study the controllability of the system when the control $u(t)$ is given by a sum of delay feedbacks

$$u(t) = \sum_{r=1}^{m} A_r x(t-h_r), \quad 0 < h_1 < \ldots < h_m.$$  

This controllability leads to the following problem:

For the delay system

$$\frac{d}{dt} x(t) = Ax(t) + \sum_{r=1}^{m} A_r x(t-h_r),$$

$$x(0) = x_0, \quad x(s) = g(s) \quad s \in [-h_m, 0), (1.1)$$

does the reachable set (with respect to initial value $x_0$ and initial function $g$) fill $X$ or become a proper subset of $X$?

This problem is called pointwise completeness or pointwise degeneracy. The problem was first proposed by Weiss [18] in his study of controllability for retarded systems in Euclidean spaces.

In case of $X = \mathbb{R}^n$, the pointwise completeness was investigated by several
authors Brooks and Schmidt [3], Zmood and MaClamroch [19] and Zverkin [20] for autonomous and nonautonomous single delayed systems.

It was Popov [13] who gave an elegant answer of pointwise degeneracy for autonomous single delayed systems in $\mathbb{R}^n$. His result was extended by Asner and Halanay [1] for autonomous systems with multiple commensurable delays. For such systems Charrier and Haugazeau [5] gave another extensions which depend on linear operator theory. Kappel [11] also obtained similar results for general nonautonomous retarded systems and gave a further analysis of systems with commensurable delays. For systems of neutral type in $\mathbb{R}^n$, the problem was solved by Choudhury [6] and Asner and Halanay [2].

The only one paper which studies the problem in infinite dimensional space is Charrier [4]. The paper gives an example of pointwise degenerate system in a Hilbert space but does not give any detailed study as in [1-3,5,6,11,13, 19].

The purpose of this paper is to develop a general theory for pointwise completeness and degeneracy of (1.1) in infinite dimensional (Banach) spaces.

We employ the delay system (1.1) studied by Datko [8]. In Section 2, we give a definition of exact and approximate pointwise completeness by taking into account of $X$ being infinite dimensional. A necessary and sufficient condition and a negative result for exact pointwise completeness are established in Section 3. Section 4 studies approximate pointwise completeness and pointwise degeneracy as a complementary concept. A main theorem is contained in Section 4. In Section 5 we specify the results of Section 4 to the systems with commensurable delays. In specifying such results the representation of fundamental solution of (1.1) given by Nakagiri [14] is effectively used.
2. System Description and Definition

Let $X$ be a reflexive Banach space with norm $\| \cdot \|$. Consider the differential system with multiple delays

$$\frac{d}{dt} x(t) = Ax(t) + \sum_{r=1}^{m} A_r x(t-h_r), \quad t > 0,$$

S:

$$x(0) = x_0, \quad x(s) = g(s) \quad s \in [-h, 0).$$  \hspace{1cm} (2.1) \hspace{1cm} (2.2)

Here we assume that $0 < h_1 < \cdots < h_m = h$ are positive constants, $x(t)$, $g(t) \in X$, operators $A_r$ ($r = 1, \cdots, m$) are bounded on $X$ and $A$ generates a strongly continuous semi-group $T(t)$, $t \geq 0$ on $X$.

In the system $S$, $x_0 \in X$ and $g$ are called an initial value and an initial function, respectively.

Under the above assumptions, R. Datko [8] has constructed the fundamental solution $G(t)$ of the system $S$ by the delay perturbation of $T(t)$.

That is, $G(t)$ satisfies the following relations:

i) $G(t) = 0$ (the null operator on $X$) if $t < 0$. \hspace{1cm} (2.3)

ii) $G(t)$ is strongly continuous on $R^+$ and satisfies

$$G(t) = T(t) + \sum_{r=1}^{m} \int_{0}^{t} T(t-s) A_r G(s-h_r) ds \quad \text{if} \quad t \geq 0.$$  \hspace{1cm} (2.4)

Let $x_0 \in X$ and $g \in L_p (-h, 0; X)$, $p \in [1, \infty]$. Then the function

$$x(t) = G(t)x_0 + \sum_{r=1}^{m} \int_{-h_r}^{0} G(t-h_r-s) A_r g(s) ds$$  \hspace{1cm} (2.5)

makes sense, the integrals being Bochner integrals in $X$, and is strongly continuous on $R^+$. Since $x(t)$ depends on $x_0$ and $g$, we denote this by $x(t; x_0, g)$. It is proved in [8] that $x(t; x_0, g)$ satisfies the integrated form of (2.1) and (2.2), i.e.,
\[ x(t; x_0, g) = T(t)x_0 + \int_0^t (T(t-s) \sum_{r=1}^m A_r x(s-h_r; x_0, g)) ds \]

if \( t \geq 0 \).  \( (2.6) \)

In this sense this function \( x(t; x_0, g) \) is called the mild solution of \( S \).

In what follows we study pointwise completeness and pointwise degeneracy by means of the mild solutions.

To give a definition of pointwise completeness, the following set of reachability is needed.

\[ R_t(L_p) = \{ x \in X : x = x(t; x_0, g) \quad \text{where} \quad x_0 \in X \quad \text{and} \quad g \in L_p(-h, 0; X) \}. \]

**DEFINITION 2.1.** The system \( S \) is said to be

(i) \( L_p \)-exactly pointwise complete at time \( t \) if \( R_t(L_p) = X \);

(ii) \( L_p \) pointwise complete at time \( t \) if \( R_t(L_p) = X \).

3. Exact Pointwise Completeness

In this section we study exact pointwise completeness.

For Banach spaces \( X, W \) and a densely defined linear operator \( L : D(L) \subset W + X \), we denote their dual Banach spaces by \( X^*, W^* \) and its adjoint operator by \( L^* \), respectively. The following abstract result is used to derive an equivalent condition for exact pointwise completeness.

**Lemma 3.1.** Let \( X \) and \( W \) be reflexive Banach spaces and let \( L \) be a bounded linear operator from \( W \) into \( X \). Then the image of \( L \) fills \( X \) if and only if there exists \( K > 0 \) such that

\[ \| x^* \|_{X^*} \leq K \| L^* x^* \|_{W^*} \]

for all \( x^* \in X^* \).  \( (3.1) \)

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This lemma follows from inverse mapping theorem [15, p.83] (see also [7]).

**Lemma 3.2.** \( G^*(t) = G(t)^* \) is strongly continuous on \( R^+ \).

**Proof.** Since \( X \) is reflexive, the weak topology of \( X \) coincides with the weak* topology. Then \( T^*(t) = T(t)^* \) is weakly continuous on \( R^+ \). From the property \( T^*(t+s) = T^*(t)T^*(s) \) and a well known result [10, p.306] that \( T^*(t) \) is strongly continuous on \( R^+ \). By (2.3)

\[
G^*(t) = T^*(t) \text{ is strongly continuous on } [0, h_1],
\]

so that

\[
G^*(t) = T^*(t) + \int_0^t T^*(s-h_1)A_sT^*(t-s)ds
\]

is also strongly continuous on \([h_1, h_2]\). Hence \( G^*(t) \) is strongly continuous on \([0, h_2]\). Continuing this procedure it is verified by (2.4) that \( G^*(t) \) is strongly continuous on \( R^+ \).

Now we can give an equivalent condition for exact pointwise completeness.

**Theorem 3.1.** Let \( p \in (1, \infty) \). Then the system \( S \) is \( p \)-exactly pointwise complete at time \( t \) if and only if there exists \( K_t > 0 \) such that

\[
\| x^* \|_{X^*} \leq K_t \max \{ \| G^*(t)x^* \|_{X^*}, \left( \int_{-h}^0 \| F^*(s)x^* \|_{X^*}^q ds \right)^{1/q} \} \quad \text{for all } x^* \in X^*,
\]  

(3.2)

where \( 1/p + 1/q = 1 \). Here the operator \( F^*_t(s) \) is given by

\[
F^*_t(s) = \sum_{r=1}^m \chi_r A^*_r G^*(t-h_r-s) \quad \text{for all } s \in [-h, 0]
\]  

(3.3)

and \( \chi_r \) is the characteristic function of \([-h_r, 0] \) (\( r = 1, \ldots, m \)).

**Proof.** Let \( W \) be the direct sum of \( X \) and \( L_p(-h, 0; X) \) whose norm \( \| \cdot \|_W \).
is introduced by

\[ \|(x,g)\|_W = \|x\| + \|g(\cdot)\|_{L_p(-h,0;X)}. \]

We denote this Banach space \( W \) by \( X \oplus L_p(-h,0;X) \). Let the operator \( L_t : X \oplus L_p(-h,0;X) \to X \) be given by

\[ L_t(x,g) = G(t)x + G_t g(\cdot) \quad \text{for} \quad (x,g) \in X \oplus L_p(-h,0;X). \tag{3.4} \]

Here \( G_t : L_p(-h,0;X) \to X \) is given by

\[ G_t g(\cdot) = \sum \int_0^h G(t-h-r)A_r g(s)ds \quad \text{for} \quad g \in L_p(-h,0;X). \tag{3.5} \]

It is evident that \( L_t \) is linear and bounded. By (3.4) we have

\[ L_t(X \oplus L_p(-h,0;X)) = R_t(L_p). \]

Then \( L_p \)-exact pointwise completeness at time \( t \) is equivalent to

\[ L_t(X \oplus L_p(-h,0;X)) = X. \]

Since \( \frac{p}{p} \in (1,\infty) \), \( W = X \oplus L_p(-h,0;X) \) is reflexive and \( W^* \) is represented by \( W^* = X^* \oplus L_q(-h,0;X^*) \) \((1/p + 1/q = 1)\) whose norm \( \|\cdot\|_{W^*} \) is given by

\[ \|(x^*,g^*)\|_{W^*} = \text{Max} \{ \|x^*\|_{X^*}, \|g^*(\cdot)\|_{L_q(-h,0;X^*)} \}. \tag{3.6} \]

To apply Lemma 3.1, we shall calculate \( L_t^* \).

For \( x^* \in X^* \) we have

\[ \langle L_t(x,g), x^* \rangle = \langle G(t)x,x^* \rangle + \sum \int_0^h G(t-h-r)A_r g(s)ds, x^* \rangle \]

\[ = \langle x, G^*(t)x^* \rangle + \int_{-h}^0 \langle g, F_t(s)x^* \rangle ds \]

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\[ = \langle x, g \rangle, L_t^*x^* \rangle_{W, W^*}, \]

where \( F_t^*(s) \) is given by (3.3). Hence \( L_t^*x^* \) is expressed by \((G^*(t)x^*, G_t^*x^*)\) and \( G_t^*x^* \) is given by

\[ \langle g, G_t^*x^* \rangle_{L_p(-h, 0; X)}, L_q(-h, 0; X^*) = \int_{-h}^{0} \langle g(s), F_t^*(s)x^* \rangle ds \]

for each \( g \in L_p(-h, 0; X). \)

Since \( G^*(t) \) is strongly continuous (Lemma 3.2), \( F_t^*(s) \) is strongly continuous on \([-h, 0]\) except for \( s = h_r, r = 1, \ldots, m-1. \) Then it follows by (3.6) that

\[ \| L_t^*x^* \| = \max \{ \| G^*(t)x^* \|, \left( \int_{-h}^{0} \| F_t^*(s)x^* \|_q^{1/q} ds \right)^{1/q} \} < \infty. \]  

(3.7)

Then applying Lemma 3.1 with \( W = X \oplus L_p(-h, 0; X) \) and \( L = L_t \), we obtain condition (3.3) for \( L_p \)-exact pointwise completeness from (3.7).

Since \( G(t) = T(t) \) for \( t \in [0, h_1] \), we have the following corollary.

**COROLLARY 3.1.** If \( T(t) \) is a strongly continuous group, then \( S \) is \( L_p \)-exactly pointwise complete at any time \( t \in [0, h_1] \) for each \( p \in (1, \infty) \).

Next we are concerned with a negative result for exact pointwise completeness. Such a fact for mild solutions in non-delay systems is first pointed out by Kuperman and Lepin [12] and more detailed researches are given by Triggiani [16,17]. In these works some types of compactness of operators are assumed to show such a fact called the lack of exact controllability. We shall show that a similar situation, which is called the lack of exact pointwise completeness, can be viewed for our delay system \( S \).
Lemma 3.3. Let $p \in (1, \infty]$ and $T(t)$ be compact for all $t > 0$. Then $G(t)$ and $G_t$ are compact for all $t > 0$.

Proof. It is easily verified by (2.4) that $T(t)$ is compact for all $t > 0$ if and only if $G(t)$ is compact for all $t > 0$. By (3.5) and changes of integral variables the compactness of $G_t$ follows from those of $G^r_t : L_p (0, h_r ; X)$ + $X$ given by

$$G^r_t g(\cdot) = \int_0^{h_r} G(t-s)A_r g(s) ds \quad \text{for } g \in L_p (0, h_r ; X) \quad (3.8)$$

for each $t > 0$ and all $r = 1, \ldots, m$. It is not difficult to prove the compactness of $G^r_t$. We remark that this lemma does not hold if $p = 1$.

THEOREM 3.2. Let $X$ be infinite dimensional and let $p \in (1, \infty]$. If $T(t)$ is compact for all $t > 0$, then $S$ is never $L_p$-exact pointwise complete at any time $t > 0$.

Proof. We shall prove this theorem by Baire category theorem as in Triggiani [16,17]. Let $R_{nm} = L_n^m (S_m)$, where $S_m$ is the closed ball in $X \oplus L_p (-h, 0; X)$ of radius $m$ with center the origin $(0,0)$. Since $L_t(x,g) = G(t)x + G_t g(\cdot)$, $L_t$ is compact for all $t > 0$ by Lemma 3.3. Hence $\overline{R_{nm}}$ is compact in X for each $n, m = 1,2, \ldots$. Since $X$ is infinite dimensional, $\overline{R_{nm}}$ cannot contain any open ball and hence is nowhere dense in $X$. This implies by Baire category theorem [15,p.80] that

$$x - \cup \{ \overline{R_{nm}} : n, m = 1,2, \ldots \} \neq \emptyset.$$

Since

$$\cup_{t>0} R_t (L_p) = \cup_{n=1}^\infty \cup_{m=1}^\infty \cup \{ \overline{R_{nm}} : n, m = 1,2, \ldots \},$$

$S$ is never $L_p$-exactly pointwise complete at any time $t > 0.$
4. Pointwise Completeness and Pointwise Degeneracy

It follows by Definition 2.1 and Hahn-Banach theorem that \( S \) is not \( L_p \)
pointwise complete at time \( t \) if and only if there exists \( x^* \neq 0 \) in \( X^* \) such
that \( x^* \in R_t^\perp (L_p) \), i.e.,
\[
\langle x, x^* \rangle = 0 \quad \text{for all} \quad x \in R_t^\perp (L_p).
\]
In this case \( S \) is said to be \( L_p \) pointwise degenerate at time \( t \) with respect
to \( x^* \). If \( S \) is \( L_p \) pointwise degenerate at time \( t \) with respect to every
\( x^* \in E^* \) for \( E^* \subset X^* \), \( S \) is called \( L_p \) pointwise degenerate at time \( t \) with
respect to \( E^* \).

The following lemma is fundamental in the arguments below.

**Lemma 4.1.** Let \( X \) and \( W \) be Banach spaces and let \( L \) be a densely defined
linear operator from \( W \) into \( X \). Then
\[
\text{Ker } L^* = (\text{Range } L)^\perp.
\]
Especially, \( (\text{Range } L) = X \) if and only if \( \text{Ker } L^* = \{0\} \) in \( X^* \).

By lemma 4.1, we obtain the following result.

**Theorem 4.1.** Let \( p \in [1, \infty) \). The system \( S \) is \( L_p \) pointwise degenerate at
time \( t > 0 \) with respect to \( E^* \) if and only if
\[
E^* \subset \text{Ker } G^*(t) \cap \bigcap \{ \text{Ker } F_t^*(s) : s \in (-h, 0) \setminus \{-h_1, \ldots, -h_m\} \big\}. \tag{4.1}
\]
Moreover, the system \( S \) is \( L_p \) pointwise complete at time \( t > 0 \) if and only if
\[
\{0\} = \text{Ker } G^*(t) \cap \bigcap \{ \text{Ker } F_t^*(s) : s \in (-h, 0) \setminus \{-h_1, \ldots, -h_m\} \big\}. \tag{4.2}
\]

Proof. By Lemma 4.1, the system \( S \) is \( L_p \) pointwise degenerate at time \( t \)
with respect to \( E^* \) if and only if
\[
E^* \subset R_t^\perp (L_p)^\perp = (\text{Range } L_t)^\perp = \text{Ker } L_t^*.
\]
Then by (3.6) it follows that

$$G^*(t)x^* = 0 \quad \text{in } X^* \quad \text{for all } x^* \in E^*$$

and

$$G^*_t x^* = 0 \quad \text{in } L^q(-h,0; X^*) \quad \text{for all } x^* \in E^*,$$

where $1/p + 1/q = 1$ and $q \neq 1$. Hence if $x^* \in E^*$,

$$\langle g, G^*_t x^* \rangle_{L^p(-h,0; X)}, L^q(-h,0; X^*) = \int_{-h}^0 \langle g(s), F^*_t(s)x^* \rangle ds = 0$$

for all $g \in L^p(-h,0; X)$.

This implies

$$F^*_t(s)x^* = 0 \quad \text{in } X^* \quad \text{for a.e. } s \in [-h, 0]. \quad (4.3)$$

Since $F^*_t(s)$ is strongly continuous on $[-h, 0]$ except for $s = h_i$, $i = 1, \ldots, m-1$, we have by (4.3) that

$$F^*_t(s)x^* = 0 \quad \text{in } X^* \quad \text{for all } s \in (-h, 0) \setminus \{-h_1, \ldots, -h_m\}.$$

Thus $x^*$ belongs to the left hand side of (4.1).

The latter part of this theorem will be clear.

The condition (4.1) is equivalent to that for all $x^* \in E^*$,

$$G^*(t)x^* = 0 \quad \text{in } X^* \quad \text{and} \quad F^*_t(s)x^* = 0 \quad \text{in } X^*$$

for all $s \in (-h, 0) \setminus \{-h_1, \ldots, -h_m\}$

(4.4)

Since (4.1) and (4.2) do not depend on the space of initial functions $L^p(-h,0; X)$, we omit $L^p$ in the terminology of $L^p$ pointwise completeness and $L^p$ pointwise degeneracy.

We now write the second condition in (4.4) by
\[ <x, \mathcal{F}_t(s)x^* > = 0 \quad \text{for all } x \in X \text{ and all } s \in (-h_{j+1}, -h_j) \]
\[ j = 0, 1, \ldots, m-1. \quad (4.5) \]

It then follows by (3.3) and changes of integral variables that (4.5) is equivalent to

\[ \sum_{r=j+1}^{m} \int_{-h_r}^{0} G(t-s-h_r)A_r x, x^* \, ds \, = \, 0 \quad \text{for all } x \in X \text{ and all } s \in [h_j, h_{j+1}], \]
\[ j = 0, 1, \ldots, m-1. \quad (4.6) \]

**Lemma 4.2.** The fundamental solution \( G(t) \) of \( S \) satisfies the following relation:

\[ G(t+s) = G(t)G(s) + \sum_{r=1}^{m} \int_{-h_r}^{0} G(t-s-h_r)A_r G(s) \, ds \quad \text{for all } s, t \geq 0. \quad (4.7) \]

Proof. Since this lemma follows easily by direct substitutions, we omit its proof.

The following lemma is due to R. Datko [9, Lemma 2.4].

**Lemma 4.3.** If \( x \in D(A) \), then \( G(t)x \) is strongly differentiable for almost everywhere on \( \mathbb{R}^+ \) and satisfies

\[ \frac{d}{dt} G(t)x = AG(t)x + \sum_{r=1}^{m} A_r G(t-h_r)x \]
\[ = G(t)Ax + \sum_{r=1}^{m} G(t-h_r)A_r x \quad \text{for a.e. } t \in \mathbb{R}^+. \quad (4.8) \]

The following theorem gives an infinite dimensional version of the results by Zmood and MaClamrock [19, Theorem 2] and Kappel [11, Theorem 2.1].
THEOREM 4.2. The system $S$ is pointwise degenerate at time $t > 0$ with respect to $E^*$ if and only if

\[ E^* \subseteq \bigcap_{s \geq t} \ker G^*(s). \quad (4.9) \]

Moreover, the system $S$ is pointwise complete at time $t > 0$ if and only if

\[ \{0\} = \bigcap_{s \geq t} \ker G^*(s). \quad (4.10) \]

Proof. It is sufficient to prove this theorem that

\[ x^* \in \bigcap_{s \geq t} \ker G^*(s) \quad (4.11) \]

if and only if

\[ x^* \in \ker G^*(t) \cap \left\{ \bigcap_{s \in (-h, 0) \setminus \{-h_1, \ldots, -h_m}\} \ker F^*_t(s) : s \in (-h, 0) \right\}. \quad (4.12) \]

The condition (4.11) is equivalent to that

\[ \langle x, G^*(s)x^* \rangle = \langle G(s)x, x^* \rangle = 0 \quad \text{for all } x \in X \text{ and all } s \geq t. \quad (4.13) \]

First we shall show that (4.12) implies (4.11). Let (4.12) be satisfied.

Then by Lemma 4.2, we have

\[ \langle G(t+s)x, x^* \rangle = \langle G(t)G(s)x, x^* \rangle + \int_{-h}^{0} \langle F^*_t(\sigma)G(\sigma+s)x, x^* \rangle d\sigma \]

\[ = \int_{-h}^{0} \langle G(\sigma+s)x, F^*_t(\sigma)x^* \rangle d\sigma = 0 \quad \text{for all } x \in X \text{ and } s \geq 0. \]

This shows (4.11).

Conversely, let (4.11) be satisfied. Then by Lemma 4.2 and (4.13), we have

\[ \text{--12--} \]
\[
\begin{align*}
\text{If } x \in D(A), \text{ by Lemma 4.3 we can differentiate } f(s, x) \text{ for a.e. } s \in \mathbb{R}^+ \text{ and obtain that}
\end{align*}
\]
\[ I_{3,j}(s) = \langle G(t+s-h_j)A_jx,x^* \rangle - \langle G(t)G(s-h_j)A_jx,x^* \rangle = 0. \]

Hence \( I_{3,j}(s) = 0 \) for all \( s \geq 0 \) and \( j = 1, \ldots, m \). Then by (4.5),
\[ I_1(s) = 0 \quad \text{for a.e.} \quad s \in \mathbb{R}^+. \]
Since \( I_1(s) \) is piecewise continuous and \( t+s-h_j \geq t \) if \( s \geq h_j \), it follows by (4.13) that
\[ I_1(s) = \sum_{r=j+1}^{m} \langle G(t+s-h_r)A_rx,x^* \rangle = 0 \quad \text{for} \quad s \in [h_j, h_{j+1}]. \]

Since \( D(A) \) is dense in \( X \), (4.6) holds. This proves that (4.12) implies (4.11) and completes the proof.

It follows from Theorem 4.2 that \( S \) is pointwise degenerate at all \( t \geq t_0 \) with respect to \( E^* \) if \( S \) is pointwise degenerate at time \( t_0 \) with respect to \( E^* \).

By Theorem 4.2, we shall call \( \cap \text{Ker} \ G^*(s) \) the degenerate space of \( S \) at time \( t \). It is clear that the degenerate space is a closed subspace of \( X^* \).

**Example 4.1.** Let \( X = L_2(0,1) \). We define the semi-group \( T(t) \) by
\[
T(t)f = g; \quad g(s) = \begin{cases} 
    f(s+t) & \text{if} \quad 0 \leq s+t \leq 1 \\
    0 & \text{if} \quad s+t > 1.
\end{cases}
\]

It is easy to verify that \( T(t) \) is a strongly continuous semi-group on \( L_2(0,1) \), \( T(t) = 0 \) for all \( t > 1 \) and its infinitesimal generator \( A = d/dt \).

We now consider the delay system
\[
\frac{d}{dt} x(t) = Ax(t) + A_1x(t-1). \tag{4.16}
\]

If \( A_1 = I \) (the identity operator on \( X \)), then the system (4.16) is pointwise complete at all \( t \geq 0 \). Next consider the case where \( A_1 = T(1/2) \). In this case the system (4.16) is pointwise degenerate at all \( t > 0 \) and the degenerate
space of the system at time $t > 0$ is given by

$$
\begin{cases}
  g^* \in L_2(0, 1): g^* = \chi_{[1-t, 1]} x & \text{for } x \in L_2(0, 1) \text{ if } 0 < t \leq 1/2 \\
  g^* \in L_2(0, 1): g^* = \chi_{[1/2, 1]} x & \text{for } x \in L_2(0, 1) \text{ if } 1/2 \leq t \leq 1 \\
  g^* \in L_2(0, 1): g^* = \chi_{[3/2 - t, 1]} x & \text{for } x \in L_2(0, 1) \text{ if } 1 \leq t < 3/2 \\
  x^* \text{ if } t \geq 3/2,
\end{cases}
$$

where $\chi_I$ is the characteristic function of $I$. Note that the degenerate space is infinite dimensional for all $t > 0$.

**Example 4.2.** (Extended Charrier's example [4]) We consider the single delay system (4.16) on a general Hilbert space $X$. The inner product and the norm are denoted by $< , >_X$ and $\| \|_X$, respectively. $A$ in (4.16) is assumed to generate a semi-group $T(t)$ on $X$. Let $E^* = \overline{\text{sp} \{ x_1^*, \cdots, x_n^* \}}$ and $\{ x_1^*, \cdots, x_n^* \} \subset \text{D}(A^*)$. We assume that there exists a set $\{ y_1, \cdots, y_n \} \subset \text{D}(A)$ such that

$$
<y_k, x_j^*>_X = \delta_{k,j}, \quad <T(1)y_k, x_j^*>_X = 0 \quad \text{and} \quad <T(1)Ay_k, x_j^*>_X = 0
$$

for all $k, j = 1, \cdots, n$.

Then if $A_1$ is given by

$$
A_1 x = \sum_{k=1}^{n} \{ <T(1)x, x_k^*>_X A y_k - <T(1)x, A x_k^*>_X y_k \},
$$

the system (4.16) is pointwise degenerate at time 2 with respect to $E^*$.

It is possible to extend this example to the case where $E^*$ is spanned by infinitely many elements $x_1^*, \cdots, x_n^*, \cdots$ in $X^*$.
5. Systems with Commensurable Delays

In this section we consider the system $S$ with commensurable delays $h_r = r, r = 1, \ldots, m, \tau > 0$.

To give a useful formula of $G(t)$ in this special system, we need some preparation. First we define the index sets $\Lambda(j,k)$ for all $j, k = 1, 2, \ldots$ by

$$\Lambda(j,k) = \{ (i_1, \ldots, i_j) : 1 \leq i_1, \ldots, i_j \leq m \text{ and } i_1 + \ldots + i_j = k \}.$$ 

Next we define the operators $T_k(t), k = 1, 2, \ldots$, by

$$T_1(t) = T(t)$$

$$T_k(t) = \sum_{j=1}^{k-1} \sum_{A(j,k-1)} T(t-s_{j-1}) A_{i_1} \ldots T(s_{j-1}) A_{i_j} t \int_{0}^{s_1} T(s) ds ds_1 \ldots ds_{j-1}, \quad k \geq 2. \quad (5.1)$$

Then $T_2(t) = \int_{0}^{t} T(t-s) A_1 T(s) ds$, for example.

For each natural number $i$, we define the matrix of operators $T_i$ by

$$T_i(t) = \begin{pmatrix}
T(t) & 0 & \cdots & 0 \\
T_2(t) T(t) & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
T_i(t) & \cdots & T_2(t) & T(t)
\end{pmatrix} \quad (5.2)$$

where $T_i(t)$ are given by (5.1).

We denote the transpose of $i$ direct sum of $X, (X \oplus X \oplus \cdots \oplus X)^t$, by $X^i$.

**Lemma 5.1.** $T_i(t)$ is a strongly continuous semigroup on $X^i$ such that its infinitesimal generator $A_i$ is represented by
\[
A_i = \begin{pmatrix}
A & O & \cdots & \cdots & \cdots & O \\
A & A & O & & & \\
& A & \ddots & \ddots & \ddots & \\
& & \ddots & A & O & \\
& & & \ddots & A & \\
& & & & \ddots & A & \\
O & \cdots & O & A & A & \cdots & A & A
\end{pmatrix}
\]

(5.3)

on \(D(A)^i = (D(A) \oplus D(A) \oplus \cdots \oplus D(A))^t\).

Lemma 5.1 says that \(D(A_i^k) = D(A)^i\). But we can not give such representations of \(D(A_i^k)\) in terms of the domains of operators \(A^k, A^{k-1}A\), etc. for \(k \geq 2\).

We define \(D_r^k \subset X\) by \(D_r^k = \bigcap_{j=1}^{k-1} D(A^{k-j}A^{j-1})\) for \(r = 1, \ldots, m\) and \(k = 1, 2, \ldots\). Then we have the following lemma.

**Lemma 5.2.**

\[
D(A_i^k) = \begin{cases}
D(A^k) \cap D_1^k \cap \cdots \cap D_{\min(i,m)}^k \\
D(A_{i-1}^k)
\end{cases}
\]

\[
\begin{cases}
D(A^k) \cap D_1^k \cap \cdots \cap D_{\min(i,m)}^k \\
\vdots \\
D(A^k) \cap D_1^k \\
D(A^k)
\end{cases}
\]

(5.4)

Since the proofs of Lemma 5.1 and Lemma 5.2 are complicated and tiresome, we omit the proofs. The following lemma is an easy consequence of (5.1) and (5.2).

**Lemma 5.3.** If \(T(t)\) is analytic, then \(T_i(t)\) is also analytic for each \(i = 1, 2, \ldots\).
Now we define $Z_i$ inductively by

$$Z_1 = I \quad \text{and} \quad Z_i = \begin{pmatrix} I & \left( T_{i-1}(\tau) Z_{i-1} \right) \\ \vdots & \vdots \end{pmatrix} \quad \text{for } i \geq 2.$$  \hspace{1cm} (5.5)

It is shown in Nakagiri [14] that $G(t)$ is represented by

$$G(t) = I \cdot T_i \cdot (t-(i-1)\tau) Z_i \quad \text{when } t \in [(i-1)\tau, i\tau),$$

where $I_i = [0, \ldots, 0, 1]$.

Let $X_{i1}$ be the largest subspace of $X$ such that $Z_{i1} \subseteq \cap_{n=0}^{\infty} D(A_{i1}^n)$. It then follows by Lemma 5.2 that if

$$\cap_{n=0}^{\infty} (D(A_{i1}^k) \cap D_{i1}^k \cap \ldots \cap D_{i1}^k)_{k=0}$$

is dense in $X$, then $X_{i1}$ is also dense in $X$.

The following theorem is a consequence from Theorem 4.2 and (5.6).

**THEOREM 5.1.** The system $S$ is pointwise degenerate at time $t_0 \in [(k-1)\tau, k\tau]$ with respect to $E^*$ if

$$E^* \subseteq \cap_{i=k}^{\infty} \cap_{n=0}^{\infty} \text{Ker} Z_i^* (A_i^*)^n I_i^*.$$ \hspace{1cm} (5.7)

Moreover, the system $S$ is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$ if

$$\cap_{i=k}^{\infty} \cap_{n=0}^{\infty} \text{Ker} Z_i^* (A_i^*)^n I_i^* = \{0\}.$$ \hspace{1cm} (5.8)

Especially for the pointwise completeness, we obtain the next theorem.

**THEOREM 5.2.** The system $S$ is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$ if

$$X = \text{sp} \{ I_i A_{i1}^n T_i X_{i1} : i = k, k+1, \ldots, n = 0, 1, 2, \ldots \}$$ \hspace{1cm} (5.9)
or, more generally, if

\[ X = \overline{\text{sp}} \left\{ I^n_{1_1 1_1} (s_i) Z_i X_i : i = k, k+1, \ldots, n = 0, 1, 2, \ldots \right\}, \]

\[ s_i \text{ arbitrary in } J_i, \]

(5.10)

where \( J_k = [t_0 - (k-1)\tau, \tau) \) and \( J_i = [0, \tau) \) for \( i \geq k+1. \)

Conversely if \( T(t) \) is analytic, \( X_i \) is dense in \( X \) for each \( i = k, k+1, \ldots \) and the system \( S \) is pointwise complete at time \( t_0 \in [(k-1)\tau, k\tau) \), then

\[ X = \overline{\text{sp}} \left\{ I^n_{1_1 1_1} (s_i) Z_i X_i : i = k, k+1, \ldots, n = 0, 1, 2, \ldots \right\}, \]

\[ s_i \text{ arbitrary in } J_i - \{0\}. \]

(5.11)

Remark 5.1. The condition (5.7) ((5.9)) is not necessary for pointwise degeneracy (pointwise completeness) in general. But if \( X_i = X \) for all \( i = k, k+1, \ldots \), i.e., \( A \) is bounded, (5.7) ((5.9)) is necessary and sufficient.

**COROLLARY 5.1.** Let \( A \) be bounded. Then \( S \) is pointwise degenerate at time \( t_0 \in [(k-1)\tau, k\tau) \) with respect to \( E^* \) if and only if (5.7) holds or

\[ E^* \subset (\overline{\text{sp}} \left\{ I^n_{1_1 1_1} Z_i X : i = k, k+1, \ldots, n = 0, 1, 2, \ldots \right\})^\perp. \]

(5.12)

Furthermore, \( S \) is pointwise complete at time \( t_0 \in [(k-1)\tau, k\tau) \) if and only if (5.8) holds or

\[ X = \overline{\text{sp}} \left\{ I^n_{1_1 1_1} Z_i X_i : i = k, k+1, \ldots, n = 0, 1, 2, \ldots \right\}. \]

(5.13)

To give a generalization of rank condition for pointwise completeness, we consider the condition (4.2). First we give the representation of \( F_t(s) \) without using characteristic functions. Let \( t \in [(k-1)\tau, k\tau) \) be fixed. Let \( X_{i,r} (i = 1, 2, \ldots, r = 1, \ldots, m) \) be the largest subspace of \( X \) such that

\[ Z^n_{1_1} X_{i,r} \subset \bigcap_{n=0}^{\infty} D(A^n_{1_1}). \]
For negative integers $i = -1, -2, \cdots$ we put $X_{i,r}^0 = X$ for each $r = 1, \cdots, m$. We now define $X_i^0, X_i^1$ by

$$X_i^0 = \bigcap_{r=i-1}^{m} X_{k+i-r, r}, \quad X_i^1 = \bigcap_{r=i}^{m} X_{k+i-r, r}$$

for $i = 1, \cdots, m$.

Then we obtain by Theorem 4.1 and differentiations of (5.6) the following result.

**Theorem 5.3.** The system $S$ is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$ if

$$X = \overline{\text{sp}} \{ G(t_0^i)X, \left( \sum_{r=i}^{m} I_{k+i-r} A^n_{k+i-r} \right) X_i^1 : i = 1, \cdots, m, n = 0, 1, 2, \cdots \}$$

(5.14)

or, more generally if

$$X = \overline{\text{sp}} \{ G(t_0^i)X, \left( \sum_{r=i}^{m} I_{k+i-r} A^n_{k+i-r} \right) X_i^0, \quad \left( \sum_{r=i}^{m} I_{k+i-r} A^n_{k+i-r} \right) X_i^1 : i = 1, \cdots, m, n = 0, 1, 2, \cdots \},$$

(5.15)

$s_i^0$ arbitrary in $(0, t_0-(k-1)\tau)$ and $s_i^1$ arbitrary in $[t_0-(k-1)\tau, \tau)$

Conversely, if $T(t)$ is analytic, $X_i^0, X_i^1$ are dense for all $i = 1, \cdots, m$ and $S$ is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$, then

$$X = \overline{\text{sp}} \{ G(t_0^i)X, \left( \sum_{r=i}^{m} I_{k+i-r} A^n_{k+i-r} \right) X_i^0, \quad \left( \sum_{r=i}^{m} I_{k+i-r} A^n_{k+i-r} \right) X_i^1 : i = 1, \cdots, m, n = 0, 1, 2, \cdots \},$$

(5.16)

$s_i^0$ arbitrary in $(0, t_0-(k-1)\tau)$ and $s_i^1$ arbitrary in $[t_0-(k-1)\tau, \tau)$.

If $A$ is bounded, then $G^*(t)$ is analytic in $t \in ((k-1)\tau, k\tau)$ for each $k = 1, 2, \cdots$. Let $t_0 \in [(k-1)\tau, k\tau)$. Then we see easily that
\[ \cap \ker G^*(t) = \cap \ker G^*(t), \]
\[ t \geq t_0 \quad t \geq (k-1)T \]
so that \( S \) is pointwise complete at \( t_0 \in [(k-1)T, kT) \) if and only if \( S \) is pointwise complete at \( t_0 = (k-1)T \). Thus we obtain the following corollary.

**Corollary 5.2.** Let \( A \) be bounded. Then \( S \) is pointwise degenerate at \( t_0 \in [(k-1)T, kT) \) with respect to \( E^* \) if and only if
\[
E^* \subset \ker Z^* A^* \cap \{ \sum_{i=1}^{m} \infty \in \ker \left( \sum_{r=i}^{m} A^* \left( \sum_{r=i}^{m} A^* \right) k^i \right) \}
\]
\[ (5.17) \]
or
\[
E^* \subset \left( \sum \left( I^m k^i \right) \right)
\]
\[ (5.18) \]
Furthermore, \( S \) is pointwise complete at \( t_0 \in [(k-1)T, kT) \) if and only if
\[
x = \sum \left( I^m k^i \right) \quad \left( \sum_{i=1}^{m} \infty \quad \sum_{r=i}^{m} A^* \left( \sum_{r=i}^{m} A^* \right) k^i \right) X \; : \; i=1, \ldots, m, \quad n=0,1,2,\ldots \}
\]
\[ (5.19) \]
or
\[
\ker Z^* A^* \cap \{ \sum_{i=1}^{m} \infty \in \ker \left( \sum_{r=i}^{m} A^* \left( \sum_{r=i}^{m} A^* \right) k^i \right) \} = \{0\}.
\]
\[ (5.20) \]
Especially if \( m = 1 \), \( (5.19) \) is reduced to
\[
x = \sum \left( I^m k^i \right) \quad \left( I_{k} A_{k} \right) X \; : \; n=0,1,2,\ldots \}
\]
\[ (5.21) \]
Consider the finite dimensional case where \( X = R^N \), \( A \) and \( A_1 \) are \( N \times N \) matrices, so that \( I_{k}, A_{k}, Z_{k} \) are \( N \times N_k \), \( N_k \times N_k \), \( N_k \times N \) matrices, respectively. In this case the condition \( (5.21) \) is equivalent to that
\[
\rank \left( I_{k} Z_{k}, I_{k} Z_{k} A_{1}, \ldots, I_{k} A_{k}^N Z_{k} A_{1} \right) = N.
\]
This fact follows by the Cayley-Hamilton theorem and gives the main result by Zmood and MacClamroch for Euclidean N-space [19,Theorem 3].
We next specify Theorem 5.2 and Theorem 5.3 in the special case where \( m = 1 \) and \( A_1 \) commutes with \( A \). Let \( X_\infty = \bigcap_{n=0}^{\infty} \text{D}(A^n) \). Clearly \( X_\infty \) is dense in \( X \).

**COROLLARY 5.3.** Let \( m = 1, A_1 \) commute with \( A \) and \( T(t) \) be analytic. Then \( S \) is pointwise degenerate at \( t_0 \in [(k-1)T, kT) \) with respect to \( E^* \) if

\[
E^* \subseteq \bigcap_{i=1}^{\infty} \bigcap_{n=0}^{\infty} \text{Ker} \ (A_1^*)^i (A^*)^n
\]

or if

\[
E^* \subseteq \text{Ker} \ G^* ((k-1)T) \cap \bigcap_{i=1}^{k} \bigcap_{n=0}^{\infty} (A_1^*)^i (A^*)^n.
\]

Furthermore, \( S \) is pointwise complete at \( t_0 \in [(k-1)T, kT) \) if

\[
X = \overline{\text{sp}} \{ A_1^n X_\infty : i = k, k+1, \ldots, n = 0, 1, 2, \ldots \}
\]

or if

\[
X = \overline{\text{sp}} \{ A_1^n X_\infty, G((k-1)T)X : i=1, \ldots, n = 0, 1, 2, \ldots \}.
\]

We now recall the definition of approximate controllability. Let \( A \) genera a semi-group \( T(t) \) on \( X \) and let \( B \) be a bounded operator from a Banach space \( U \) into \( X \). Then the system \( \{A, B\} \) is said to be approximately controllable if \( \bigcup_{t>0} T(t)BU = X \). We define \( \tilde{Z}_i \) by

\[
\tilde{Z}_1 = I \quad \text{and} \quad \tilde{Z}_i = \begin{pmatrix} I & 0 \\ 0 & T_{i-1}(T) \tilde{Z}_{i-1} \end{pmatrix} \quad \text{for } i \geq 2.
\]

The following corollary is immediate from Theorem 4.2, Lemma 5.3 and (5.6).

**COROLLARY 5.4.** Let \( T(t) \) be analytic and the system \( \{A_k, \tilde{Z}_k\} \) be approximately controllable on \( X^i \). Then \( S \) is pointwise complete at any time \( t \in [0, kT) \).

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REFERENCES


