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FINITE-DIMENSIONAL REGULATOR DESIGN FOR INFINITE-DIMENSIONAL SYSTEMS WITH CONSTANT DISTURBANCES

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In this paper we investigate a regulator problem for an infinite-dimensional system with constant disturbances. The regulator problem considered here is to determine a feedback control law which stabilizes and regulates the system. From a practical point of view we propose a design procedure of a regulator which can be realized in finite-dimensional theories and techniques for an infinite-dimensional system. In the design procedure it is necessary to construct a state observer in order to estimate the system state from observations. We present explicit sufficient conditions for the convergence of the schemes.

1. System description and problem formulation.

We consider the system described by an evolution equation on a reflexive Banach space X:

$$\frac{du(t)}{dt}=Au(t)+Bf(t)+w, \quad 0<t<t_1, \quad u(0)=u_0 \varepsilon D(A) \tag{1.1}$$

where $u(t)\varepsilon X$ is the system state vector, $f(t)\varepsilon E^p$ is the control vector and $w\varepsilon X$ is an unknown constant disturbance vector. The operator $A:D(A)\to X$ is a closed, linear, densely defined generator of a holomorphic semigroup $U(t)$ on X. The control $f(t)$ is assumed to be Hölder continuous. The operator B is a bounded linear operator from a p-dimensional Euclidean space E^p to X.

Then the system (1.1) has a unique solution $u(t)\varepsilon D(A)$ for $t\geqq 0$, continuous for $t\geqq 0$ and continuously differentiable for $t>0$, given by

$$u(t)=U(t)u_0+\int_0^t U(t-s)(Bf(s)+w)ds. \tag{1.2}$$

The controlled output $y(t)\varepsilon E^r$ is given by

(1.3) $y(t)=Cu(t)$, $0<t<t_1$

where the output operator $C:D(C)\subset X\to E^r$ is linear and defined on $D(A)$, and hence $D(A)\subset D(C)$. The operator C is assumed to be A-bounded.

The key to finite-dimensional regulator design is to a decomposition of the state space X based on the modes of the system. The operator A satisfies the spectrum decomposition assumption[6];then there exists the projection P such that

(1.4) $X=PX+(I-P)X$

and PX, (I-P)X form A invariant subspaces of X. From the viewpoints of system analysis and synthesis, it is practical and interesting to take PX as a finite-dimensional space. We shall assume henceforth that PX is the N-dimensional subspace.

Consequently from (1.1) and (1.3)

(1.5) $\frac{dPu(t)}{dt}=A_P Pu(t)+PBf(t)+Pw,\quad Pu(0)=Pu_0,$

(1.6) $\frac{dQu(t)}{dt}=A_Q Qu(t)+QBf(t)+Qw,\quad Qu(0)=Qu_0,$

(1.7) $y(t)=C_P Pu(t)+C_Q Qu(t)$

and

$$u = Pu + Qu \qquad u\varepsilon X,$$

where $Q=I-P$ and A_P, A_Q are the restrictions of A to PX and QX, respectively. PB, QB are the restrictions of B to PX and QX, restrictively. C_P, C_Q are the restrictions of C to PX and QX. $U_P(t)=PU(t)$ is generated by A_P and $U_Q(t)=QU(t)$ is generated by A_Q. Actually A_P is bounded on PX and $U_P(t)$ is a uniformly continuous holomorphic semigroup.

We also assume that the operator A_Q is a genarator of a exponentially stable semigroup $U_Q(t)$ such that for constants $K\geqq 1$ and $\sigma>0$

(1.8) $||U_Q(t)||\leqq Ke^{-\sigma t}$, $t>0$.

Now we may pose the following control problem.

PROBLEM 1.

Find a linear feedback control law for the system (1.1) and (1.3) such that (i) the resulting closed-loop system without a disturbance w will be exponentially stable

and

(ii) the controlled output y(t) will be regulated so that $y(t)\to y_d$, $t\to\infty$ independent on w where $y_d\in E^r$ represents a desired constant reference vector.

The assumption that the disturbance vector w and the reference vector are constant in time t, is not the most general. We can treat polynomial signals in time t. However, the constant vectors are most important and they allow us to develop the theory without unnecessary mathematical complexity.

2. Construction of state feedback controllers.

In this chapter we construct a feedback controller which solves Problem 1. The controller consists of two parts: the stabilizing compensator (Proportional part of the controller) and the servocompensator (Integral part of the controll -er). The role of the servocompensator is to change the system steady state, so that the output regulation $y(t)\to y_d$ will occur.

Now if we put

(2.1) $\dot{\eta}(t)=y(t)-y_d$,

we obtain from (1.1) and (1.3) the following system

(2.2) $$\begin{bmatrix}\dot{\eta}\\ \dot{u}\end{bmatrix}=\begin{bmatrix}0 & C\\ 0 & A\end{bmatrix}\begin{bmatrix}\eta\\ u\end{bmatrix}+\begin{bmatrix}0\\ B\end{bmatrix}f+\begin{bmatrix}-y_d\\ w\end{bmatrix},$$

(2.3) $$y=[0 \quad C]\begin{bmatrix}\eta\\ u\end{bmatrix}$$

in the extended state space $X_r=E^r\times X$, which will be a Banach space, when equipped with the norm

$$||\cdot||^2_{X_r}=||\cdot||^2_{E^r}+||\cdot||^2_X.$$

Before designing a compensator, it is useful to transform the state variable as follows:

$$(2.4)\qquad \begin{cases} \xi=\eta+Su \\ u=u \end{cases}$$

where S is a bounded linear operator from X to E^p. By this transformation we get from (2.2)

$$\begin{bmatrix} \dot\xi \\ \dot u \end{bmatrix}=\begin{bmatrix} 0 & C+SA \\ 0 & A \end{bmatrix}\begin{bmatrix} \xi \\ u \end{bmatrix}+\begin{bmatrix} SB \\ B \end{bmatrix}f+\begin{bmatrix} Sw-y_d \\ w \end{bmatrix},$$

that is,

$$(2.5)\qquad \begin{bmatrix} \dot\xi \\ P\dot u \\ Q\dot u \end{bmatrix}=\begin{bmatrix} 0 & C_P+S_PA_P & C_Q+S_QA_Q \\ 0 & A_P & 0 \\ 0 & 0 & A_Q \end{bmatrix}\begin{bmatrix} \xi \\ Pu \\ Qu \end{bmatrix}+\begin{bmatrix} SB \\ PB \\ QB \end{bmatrix}f+\begin{bmatrix} Sw-y_d \\ Pw \\ Qw \end{bmatrix}.$$

Since the operator A_Q is the generator of a exponentially stable semigroup $U_Q(t)$, the inverse A_Q^{-1} exists and is bounded. If we take $S_P=0$ and $S_Q=-C_QA_Q^{-1}$, the operator $S=(0,S_Q)$ is actually a bounded linear operator from X to E^p. In this case (2.5) becomes

$$(2.6)\qquad \begin{bmatrix} \dot\xi \\ P\dot u \\ Q\dot u \end{bmatrix}=\begin{bmatrix} 0 & C_P & 0 \\ 0 & A_P & 0 \\ 0 & 0 & A_Q \end{bmatrix}\begin{bmatrix} \xi \\ Pu \\ Qu \end{bmatrix}+\begin{bmatrix} S_QQB \\ PB \\ QB \end{bmatrix}f+\begin{bmatrix} S_QQw-y_d \\ Pw \\ Qw \end{bmatrix}.$$

Thus (2.2) and (2.3) become

$$(2.7)\qquad \begin{bmatrix} \dot\xi \\ \dot u \end{bmatrix}=\begin{bmatrix} 0 & \overline{C} \\ 0 & A \end{bmatrix}\begin{bmatrix} \xi \\ u \end{bmatrix}+\begin{bmatrix} SB \\ B \end{bmatrix}f+\begin{bmatrix} Sw-y_d \\ w \end{bmatrix}$$

$$(2.8)\qquad y=[0 \quad C]\begin{bmatrix} \xi \\ u \end{bmatrix}$$

where $\overline{C}=(C_P,0)$, $SB=S_QQB=-C_QA_Q^{-1}QB$ and $Sw=S_QQw=-C_QA_Q^{-1}Qw$.

For the system (2.7) we consider a linear feedback control law

$$(2.9)\qquad \begin{aligned} f(t)&=D\xi(t)+Fu(t) \\ &=(DS+F)u(t)+D\int_0^t(y(s)-y_d)ds \end{aligned}$$

where $D\in L(E^r,E^p)$ and $F\in L(X,E^p)$. Then we get the closed-loop system

$$(2.10)\qquad \begin{bmatrix} \dot\xi \\ \dot u \end{bmatrix}=\begin{bmatrix} SBD & \overline{C}+SBF \\ BD & A+BF \end{bmatrix}\begin{bmatrix} \xi \\ u \end{bmatrix}+\begin{bmatrix} Sw-y_d \\ w \end{bmatrix}$$

$$y=[0 \quad C]\begin{bmatrix} \xi \\ u \end{bmatrix}.$$

[Theorem 1]

Suppose that there is a stabilizing control $f=D\xi+Fu$ such that the system

$$(2.11)\quad \begin{bmatrix}\dot{\xi}\\ \dot{u}\end{bmatrix}=\begin{bmatrix}0 & \overline{C}\\ 0 & A\end{bmatrix}\begin{bmatrix}\xi\\ u\end{bmatrix}+\begin{bmatrix}SB\\ B\end{bmatrix}f$$

will be exponentially stable. Then regulation will occur in spite of a constant disturbance w.

Proof.

Let us introduce the following notations

$$A_0=\begin{bmatrix}0 & \overline{C}\\ 0 & A\end{bmatrix},\quad A_f=\begin{bmatrix}SBD & \overline{C}+SBF\\ BD & A+BF\end{bmatrix}.$$

The operator A_0 is a generator of a strongly continuous semigroup on X_r. The operator A_f is a bounded perturbation of A_0. Thus A_f is also a generator of a strongly continuous semigroup $U_f(t)$ on X_r. Since A_f was assumed exponentially stable, the inverse A_f^{-1} exists and is bounded.

The unique solution of (2.10) is given by

$$\begin{bmatrix}\xi(t)\\ u(t)\end{bmatrix}=U_f(t)\begin{bmatrix}\xi(0)\\ u(0)\end{bmatrix}+\int_0^t U_f(t-s)\begin{bmatrix}Sw-y_d\\ w\end{bmatrix}ds$$

$$=U_f(t)\begin{bmatrix}\xi(0)\\ u(0)\end{bmatrix}+\int_0^t U_f(t-s)A_fA_f^{-1}\begin{bmatrix}Sw-y_d\\ w\end{bmatrix}ds$$

$$=U_f(t)\begin{bmatrix}\xi(0)\\ u(0)\end{bmatrix}+U_f(t)A_f^{-1}\begin{bmatrix}Sw-y_d\\ w\end{bmatrix}-A_f^{-1}\begin{bmatrix}Sw-y_d\\ w\end{bmatrix}.$$

Letting $t\to\infty$, we have

$$\lim_{t\to\infty}\begin{bmatrix}\xi(t)\\ u(t)\end{bmatrix}=-A_f^{-1}\begin{bmatrix}Sw-y_d\\ w\end{bmatrix},$$

since $U_f(t)$ is a stable semigroup. The output is now given by

$$(2.12)\quad \lim_{t\to\infty} y(t)=-[0\quad C]A_f^{-1}\begin{bmatrix}Sw-y_d\\ w\end{bmatrix}.$$

Next consider the equation

$$(2.13)\quad A_f\begin{bmatrix}\xi^*\\ u^*\end{bmatrix}=\begin{bmatrix}Sw-y_d\\ w\end{bmatrix}$$

which implies

$$SBD\xi^*+(\bar{C}+SBF)u^*=Sw-y_d$$

$$BD\xi^*+(A+BF)u^*=w.$$

From these equations we obtain

$$SAu^*-\bar{C}u^*=y_d$$

from which we get $Cu^*=-y_d$, since $SA+C=\bar{C}$. Equations (2.12) and (2.13) imply

$$\lim_{t\to\infty} y(t)=-[0 \quad C]\begin{bmatrix}\xi^*\\ u^*\end{bmatrix}=-Cu^*=y_d$$

which proves the regulation independent of w.

Now we have proved that if the augmented system (2.11) can be stabilized, then regulation will automatically occur. The next step is to show that there exists a stabilizing control of the form $f=D\xi+Fu$ where $D\varepsilon L(E^r,E^p)$ and $F\varepsilon L(X,E^p)$. The key to stabilizability for the system (2.11) is a decomposition of the system state u on the modes of the system.

Decomposing the state u by Pu and Qu, we obtain from (2.11)

$$\begin{bmatrix}\dot{\xi}\\ P\dot{u}\\ Q\dot{u}\end{bmatrix}=\begin{bmatrix}0 & C_P & 0\\ 0 & A_P & 0\\ 0 & 0 & A_Q\end{bmatrix}\begin{bmatrix}\xi\\ Pu\\ Qu\end{bmatrix}+\begin{bmatrix}SB\\ PB\\ QB\end{bmatrix}f.$$

For this system let us consider a linear feedback control law

(2.14) $$f(t)=D\xi(t)+F_0Pu(t)$$

$$=DSu(t)+F_0Pu(t)+D\int_0^t(y(s)-y_d)ds$$

where $F_0\varepsilon L(PX,E^p)$. Then we have the closed-loop system

$$\begin{bmatrix}\dot{\xi}\\ P\dot{u}\\ Q\dot{u}\end{bmatrix}=\begin{bmatrix}SBD & C_P+SBF_0 & 0\\ PBD & A_P+PBF_0 & 0\\ QBD & QBF_0 & A_Q\end{bmatrix}\begin{bmatrix}\xi\\ Pu\\ Qu\end{bmatrix}.$$

If the (r+N) dimensional system

(2.15) $$\left(\begin{bmatrix}0 & C_P\\ 0 & A_P\end{bmatrix},\begin{bmatrix}SB\\ PB\end{bmatrix}\right)$$

is controllable, there exist feedback control operators D and F_0 such that all the eigenvalues of A_{fP}

$$A_{fP}=\begin{bmatrix} SBD & C_P+SBF_0 \\ PBD & A_P+PBF_0 \end{bmatrix}$$

have negative real parts. The feedback operators D and F_0 can be determined by pole allocation or optimal regulator design for the usual finite-dimensional system[8].

Now we have for constants $K_2\geqq 1$ and $\omega>0$

$$||\exp(A_{fP}t)||\leqq K_2e^{-\omega t}, \quad t\geqq 0.$$

This implies

$$(2.16) \qquad \left|\left|\begin{bmatrix}\xi(t)\\ Pu(t)\end{bmatrix}\right|\right|\leqq K_2e^{-\omega t}\left|\left|\begin{bmatrix}\xi(0)\\ Pu_0\end{bmatrix}\right|\right|, \quad t\geqq 0.$$

Let us estimate Qu(t). Since

$$Qu(t)=U_Q(t)Qu_0+\int_0^t U_Q(t-s)QB[D \quad F_0]\begin{bmatrix}\xi(s)\\ Pu(s)\end{bmatrix}ds,$$

from (1.8) and (2.16) we have

$$||Qu(t)||\leqq K||Qu_0||e^{-\sigma t}+\int_0^t KK_2||QB||\,||[D \quad F_0]||\,\left|\left|\begin{bmatrix}\xi(0)\\ Pu_0\end{bmatrix}\right|\right|e^{-\sigma(t-s)}e^{-\omega s}ds$$

$$=K||Qu_0||e^{-\sigma t}+KK_2||QB||\,||[D \quad F_0]||\,\left|\left|\begin{bmatrix}\xi(0)\\ Pu_0\end{bmatrix}\right|\right|\left|\frac{e^{-\omega t}-e^{-\sigma t}}{\sigma-\omega}\right|$$

where we choose ω such that $\omega\neq\sigma$. Consequently we obtain

$$(2.17) \qquad ||Qu(t)||\leqq c_1e^{-\min(\sigma,\omega)t}\left|\left|\begin{bmatrix}\xi(0)\\ u_0\end{bmatrix}\right|\right|, \quad t\geqq 0$$

where $c_1=\sqrt{2}\max(K,\sqrt{K^2||P||^2+(KK_2||QB||\,||[D \quad F_0]||\frac{1}{|\sigma-\omega|})^2||Q||^2})$.

Moreover the estimates (2.16) and (2.17) give

$$\left|\left|\begin{bmatrix}\xi(t)\\ u(t)\end{bmatrix}\right|\right|=\sqrt{||\xi(t)||^2+||Pu(t)+Qu(t)||^2}$$

$$\leqq\sqrt{2}\sqrt{K_2^2e^{-2\omega t}\left|\left|\begin{bmatrix}\xi(0)\\ Pu_0\end{bmatrix}\right|\right|^2+c_1^2e^{-2\min(\sigma,\omega)t}\left|\left|\begin{bmatrix}\xi(0)\\ u_0\end{bmatrix}\right|\right|^2}$$

$$\leqq c_2e^{-\min(\sigma,\omega)t}\left|\left|\begin{bmatrix}\xi(0)\\ u_0\end{bmatrix}\right|\right|, \quad t\geqq 0.$$

Thus we have obtained the following estimate.

(2.18) $||U_f(t)|| \leqq c_2 e^{-\min(\sigma,\omega)t}, \quad t \geqq 0$

where $c_2=\sqrt{2(K_\xi^2+c_1^2)}\max(\sqrt{2},\sqrt{1+||P||^2})$.

It has been shown that the system (2.11) is exponentially stabilized by the feedback control law $f=D\xi+Fu$ where $F=F_0P$. We have obtained the following theoerem.

[Theorem 2]

If the (r+N) dimensional system (2.15) is controllable, then there exists a feedback control law

$$f(t)=DSu(t)+Fu(t)+D\int_0^t(y(s)-y_d)ds$$

which exponentially stabilizes and regulates the system (1.1).

Remark 1.

It is easily shown the (r+N) dimensional system (2.15) is controllable if the N dimensional subsystem (A_p,PB) is controllable and

$$\operatorname{rank}\begin{bmatrix} SB & C_p \\ PB & A_p \end{bmatrix}=r+N\leqq p+N.$$

In our design schemes it is not clear how fast the output y(t) will converge to the reference vector y_d. If for the system (2.7) we apply the feedback control law (2.14), we get the closed-loop system

(2.19) $\dot{u}_\xi=A_f u_\xi(t)+w_s$

where $u_\xi(t)=\begin{bmatrix}\xi \\ u\end{bmatrix}$, $w_s=\begin{bmatrix}Sw-y_d \\ w\end{bmatrix}$. The solution is given by

$$u_\xi(t)=U_f(t)u_\xi(0)+\int_0^t U_f(t-s)w_s ds.$$

Differentiating this in t, we obtain

$$\dot{u}_\xi(t)=U_f(t)(A_f u_\xi(0)+w_s).$$

From (2.18)

$$||\dot{u}_\xi(t)|| \leqq c_2 e^{-\min(\sigma,\omega)t}||A_f u_\xi(0)+w_s||, \quad t\geqq 0.$$

Since (2.4) implies $\dot{\xi}(t)=\dot{\eta}(t)+S\dot{u}(t)=(y(t)-y_d)+S_Q Q\dot{u}(t)$, we get the estimate

(2.20) $||y(t)-y_d|| \leqq Const.e^{-\min(\sigma,\omega)t}||A_f u_\xi(0)+w_s||, \quad t\geqq 0.$

This estimate says that the controlled output $y(t)$ will converge to the constant reference vector y_d with an arbitrarily assignable exponential decay rate by the feedback controller (2.15).

3. Construction of output feedback controllers.

Theorem 1 and Theorem 2 give the basic solution for Problem 1. However we assume that the knowledge of Su and Pu in the feedback control law (2.14). In this chapter we shall show that even if we use the state v of an observer in place of u, the feedback control law (2.14) still gives the solution of Problem 1.

Consider the measurement output $z(t)$ given by

(3.1) $\quad z(t)=Mu(t), \quad t\geq 0$

where the mesurement operator $M:D(M)\to E^q$ is linear and defined on $D(A)$, and hence $D(A)\subset D(M)$. The operator M is assumed to be A-bounded.

Now we construct an identity observer

(3.2) $\quad \dot{v}(t)=Av(t)+Bf(t)-G(Mv(t)-z(t)), \quad 0<t<t_1, \quad v(0)=0.$

Here G is a compact operator from E^q to X. Then $A_G=A-GM$ generates a holomorphic semigroup $T(t)$ on X[12] and the solution (3.2) is given by

$$v(t)=\int_0^t T(t-s)(Bf(s)+Gz(s))ds.$$

The solution $v\varepsilon C(0,t_1;X)$. We can prove the following lemma.

Lemma 1.

If the N dimensional system (A_p,M_p) is observable, then there exists an operatc G such that the semigroup $T(t)$ will be exponentially stable.

Proof.

Consider the system

$$\dot{\bar{v}}=Av(t)-GM\bar{v}(t), \quad \bar{v}(0)=\bar{v}_0\varepsilon D(A).$$

We choose G such that QG=0, that is,

$$Gz=\begin{cases} G_0z & \text{on } PX \\ 0 & \text{on } QX \end{cases} \quad \text{for } z\varepsilon E^q$$

where $G_0\varepsilon L(E^q,PX)$. Decomposition $\bar{v}$ by $P\bar{v}$ and $Q\bar{v}$, we have

$$(3.3)\qquad \begin{aligned}&P\dot{\bar{v}}=A_P P\bar{v}-G_0(M_P P\bar{v}+M_Q Q\bar{v}), \quad P\bar{v}(0)=P\bar{v}_0\\ &Q\dot{\bar{v}}=A_Q Q\bar{v}, \quad Q\bar{v}(0)=Q\bar{v}_0.\end{aligned}$$

Since PX is the N dimensional subspace of V, from finite dimensional theory[8], we can find a $G_0\varepsilon L(E^q,PX)$ such that all the eigenvalues of $A_{GP}=A_P-G_0M_P$ have negative real parts. Thus there are constants $K_G\geqq 1$ and $\gamma>0$ such that

$$(3.4)\qquad ||\exp(A_{GP}t)||\leqq K_G e^{-\gamma t}, \quad t\geqq 0.$$

From (3.4)

$$P\bar{v}(t)=\exp(A_{GP}t)P\bar{v}_0-\int_0^t \exp(A_{GP}(t-s))G_0M_QQ\bar{v}(s)ds.$$

The estimates (1.8) and (3.4) imply

$$\begin{aligned}||P\bar{v}(t)||&\leqq K_G||P\bar{v}_0||e^{-\gamma t}+K_G||G_0||\int_0^t e^{-\gamma(t-s)}||M_QU_Q(\tfrac{s}{2})||\,||U_Q(\tfrac{s}{2})Q\bar{v}_0||ds\\ &\leqq K_G||P\bar{v}_0||e^{-\gamma t}+KK_G||G_0||\int_0^t e^{-\gamma(t-s)}e^{-\frac{1}{2}\sigma s}||M_QU_Q(\tfrac{s}{2})||ds\\ &\leqq K_G||P\bar{v}_0||e^{-\gamma t}+KK_G||G_0||M_QU_Q(\cdot)||(\frac{e^{-\sigma t}-e^{-2\gamma t}}{2\gamma-\sigma})^{\frac{1}{2}}||Q\bar{v}_0||\end{aligned}$$

since we can choose such that $2\gamma\neq\sigma$. Here we have assumed

$$(3.5)\qquad M_QU_Q(\cdot)\varepsilon L^2(0,t_1;E^q).$$

Consequently we have

$$\begin{aligned}||\bar{v}(t)||&\leqq||P\bar{v}(t)||+||Q\bar{v}(t)||\\ &\leqq c_3e^{-\min(\frac{\sigma}{2},\gamma)t}||\bar{v}_0||, \quad t\geqq 0\end{aligned}$$

where $c_3=\{K_G||P||+K||Q||(1+K_G||G_0||||M_QU_Q(\cdot)||\frac{}{\sqrt{|2\gamma-\sigma|}})\}$. Thus we obtain the estimate

$$(3.6)\qquad ||T(t)||\leqq c_3e^{-\min(\frac{\sigma}{2},\gamma)t}, \quad t\geqq 0.$$

From (1.1), (3.1) and (3.3) the estimated error vector e=v-u satisfies

$$(3.7)\qquad \dot{e}(t)=A_Ge(t)-w, \quad e(0)=-u_0.$$

Even if the operator A_G generates an exponentially stable semigroup T(t), there remains an estimated error in the steady state, since w is a constant vector in time t.

However we can show that the feedback control law

$$f(t)=DSv(t)+Fv+D\int_0^t (y(t)-y_d)dt \tag{3.8}$$

gives the solution for Problem 1. From (2.7), (3.7) and (3.8) we get the closed-loop system

$$\begin{bmatrix} \dot{e} \\ \dot{\xi} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A_G & 0 & 0 \\ SBDS+SBF & SBD & \overline{C}+SBF \\ BDS+BF & BD & A+BF \end{bmatrix} \begin{bmatrix} e \\ \xi \\ u \end{bmatrix} + \begin{bmatrix} -w \\ Sw-y_d \\ w \end{bmatrix} \tag{3.9}$$

$$y=[0 \quad 0 \quad C]\begin{bmatrix} e \\ \xi \\ u \end{bmatrix}$$

in the extended state space $X_q=X\times E^q\times X$, which will be a Banach space, when equipped with the norm

$$||\cdot||^2_{X_q}=||\cdot||^2_X+||\cdot||^2_{E^q}+||\cdot||^2_X.$$

Corresponding to Theorem 1 the following theorem holds.

[Theorem 3]

If the system

$$\begin{bmatrix} \dot{e} \\ \dot{\xi} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A_G & 0 & 0 \\ SBDS+SBF & SBD & \overline{C}+SBF \\ BDS+BF & BD & A+BF \end{bmatrix} \begin{bmatrix} e \\ \xi \\ u \end{bmatrix} \tag{3.10}$$

is exponentially stable, then regulation will occur in spite of a constant disturbance w.

The next step is to show that the system (3.10) will be exponentially stable, if both the semigroup T(t) generated by A_G and the semigroup $U_f(t)$ generated by A_f are exponentially stable. Introducing the notations

$$u_\xi=\begin{bmatrix} \xi \\ u \end{bmatrix}, \quad B_f=\begin{bmatrix} SBDS+SBF \\ BDS+BF \end{bmatrix},$$

we get from (3.10)

$$e(t)=T(t)e(0)$$

$$u_\xi(t)=U_f(t)u_\xi(0)+\int_0^t U_f(t-s)B_f e(s)ds.$$

Using (2.18) and (3.6) we can estimate

$$||e(t)||\leqq c_3 e^{-\min(\frac{\sigma}{2},\gamma)t}||e(0)||,\quad t\geqq 0$$

$$||u_\xi(t)||\leqq c_2 e^{-\min(\sigma,\omega)t}||u_\xi(0)||+c_2c_3||B_f||\,e(0)||\int_0^t e^{-\min(\sigma,\omega)(t-s)}$$
$$\times e^{-\min(\frac{\sigma}{2},\gamma)s}ds$$

$$=c_2 e^{-\min(\sigma,\omega)t}||u_\xi(0)||+c_2c_3||B_f||\frac{e^{-\min(\frac{\sigma}{2},\gamma)t}-e^{-\min(\sigma,\omega)t}}{\min(\sigma,\omega)-\min(\frac{\sigma}{2},\gamma)}||e(0)||$$

since we can shoose ω and γ such that $\min(\sigma,\omega)\neq\min(\frac{\sigma}{2},\gamma)$.

Thus we have

$$(3.11)\qquad ||u_\xi(t)||\leqq c_4 e^{-\min(\gamma,\omega,\frac{\sigma}{2})t}\left\|\begin{bmatrix}e(0)\\u_\xi(0)\end{bmatrix}\right\|,\quad t\geqq 0$$

where

$$c_4=\min(c_2+c_2c_3b_f,\sqrt{2}\max(c_2,c_2c_3b_f)),\quad b_f=\frac{||B_f||}{|\min(\sigma,\omega)-\min(\frac{\sigma}{2},\gamma)|}.$$

Consequently we obtain

$$(3.12)\qquad ||U_G(t)||\leqq c_5 e^{-\min(\gamma,\omega,\frac{\sigma}{2})t},\quad t\geqq 0$$

where $c_5=\sqrt{c_3^2+c_4^2}$ The semigroup $U_G(t)$ is generated by A_{Gf}

$$A_{Gf}=\begin{bmatrix}A_G & 0 & 0\\ SBDS+SBF & SBD & \overline{C}+SBF\\ BDS+BF & BD & A+BF\end{bmatrix}.$$

We can obtain the following theorem from Theorem 2 and Lemma 1.

[Theorem 4]

If the (r+N) dimensional system (2.15) is controllable and the N dimensional system (A_p,M_p) is observable, then there exists a feedback control law (3.8) which exponentially stabilizes and regulates the system (1.1).

Moreover if for the system (2.7) we apply the feedback control law (3.8), we get the closed-loop system

$$(3.13)\qquad \dot{u}_{e\xi}=A_{Gf}u_{e\xi}(t)+\overline{w}$$

where $u_{e\xi}=\begin{pmatrix} e \\ u_\xi \end{pmatrix}$, $\overline{w}=\begin{pmatrix} -w \\ w_s \end{pmatrix}$. The solution is given by

$$u_{e\xi}(t)=U_G(t)u_{e\xi}(0)+\int_0^t U_G(t-s)\overline{w}ds.$$

Differentiating this in t, we obtain

$$\dot{u}_{e\xi}(t)=U_G(t)(A_{Gf}u_{e\xi}(0)+\overline{w}).$$

From (3.12)

$$||\dot{u}_{e\xi}(t)||\leqq c_5 e^{-\min(\gamma,\omega,\frac{\sigma}{2})t}||A_{Gf}u_{e\xi}(0)+\overline{w}||, \quad t\geqq 0.$$

which implies that the estimate

$$(3.14) \qquad ||y(t)-y_d||\leqq \text{Const.}e^{-\min(\gamma,\omega,\frac{\sigma}{2})t}||A_{Gf}u_{e\xi}(0)+\overline{w}||, \quad t\geqq 0.$$

This estimate says that the controlled output y(t) will converge to the constant reference vector y_d with an arbitrarily assignable exponential decay rate by the feedback controller (3.8).

However, the infinite-dimensional observer (3.2) is not so easy to realize. We can show that the system (1.1) and (1.3) is stabilized and regulated by output feedback through a finite-dimensional observer.

Define the other projections P_L and Q_L such that

$$X=P_LX+Q_LX$$

and P_LX is the L dimensional subspace where $L\geqq N$. We construct an observer

$$(3.15) \qquad \dot{v}(t)=Av(t)+P_LBf(t)-G(Mv(t)-z(t)), \quad 0<t<t_1, \quad v(0)=0.$$

Then we get the following system corresponding to (3.10)

$$(3.16) \qquad \begin{pmatrix} \dot{e} \\ \dot{\xi} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} A_G-Q_LB(DS+F) & -Q_LBD & -Q_LBF \\ SBDS+SBF & SBD & \overline{C}+SBF \\ BDS+BF & BD & A+BF \end{pmatrix}\begin{pmatrix} e \\ \xi \\ u \end{pmatrix}.$$

This system operator is added a bounded perturbation

$$\overline{B}_L=\begin{pmatrix} -Q_LB(DS+F) & -Q_LBD & -Q_LBF \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

to A_{Gf}. The operator $A_{Gf}+\overline{B}_L$ generates a strongly continuous semigroup $\overline{U}_G(t)$, defined by

$$\overline{U}_G(t)u_q=U_G(t)u_q+\int_0^t U_G(t-s)\overline{B}_L\overline{U}_G(s)u_q ds, \quad u_q \varepsilon X_q.$$

Moreover from (3.12) we obtain the estimate [2],[6]

$$(3.17) \qquad ||\overline{U}_G(t)|| \leqq c_5 e^{(-\min(\gamma,\omega,\frac{\sigma}{2})+c_5||\overline{B}_L||)t}, \quad t \geqq 0.$$

If we choose L such that $-\min(\gamma,\omega,\frac{\sigma}{2})+c_5||\overline{B}_L|| \leqq -\delta<0$, the system (3.16) is exponentially stable. In this case the control law (3.8) still stabilizes and regulates the system (1.1) using the observer (3.15) in place of the observer (3.2).

On the other hand since $L \geqq N$, the restriction of G to $P_L X$ is G_0 and the restriction of G to $Q_L X$ is 0. The observer (3.15) is decomposed as follows.

$$\begin{cases} P_L\dot{v}(t)=(A_{PL}-G_0M_{PL})P_Lv(t)+P_LBf(t)-G_0M_{QL}Q_Lv(t)+G_0z(t) \\ Q_L\dot{v}(t)=A_{QL}Q_Lv(t), \end{cases}$$

where A_{PL} and A_{QL} are the restrictions of A to $P_L X$ and $Q_L X$, respectively. M_{PL} and M_{QL} are the restrictions of M to $P_L X$ and $Q_L X$.

We are free to choose $Q_L v(0)=0$ and this implies that $Q_L v(t)=0$, $t \geqq 0$. Thus an L dimensional compensator is given by

$$(3.18) \qquad \begin{aligned} &P_L\dot{v}(t)=(A_{PL}-G_0M_{PL})P_Lv(t)+P_LB\overline{f}(t)+G_0z(t) \\ &\overline{f}(t)=(DS+F)P_Lv(t)+D\int_0^t(y(s)-y_d)ds. \end{aligned}$$

Therefore if we choose L such that $-\min(\gamma,\omega,\frac{\sigma}{2})+c_5||B_L|| \leqq -\delta<0$, we can stabilize and regulate the system by output feedback through an L dimensional observer. For the L dimensional compensator (3.18) Theorem 3 and Theorem 4 still hold. Moreover y(t) will converge to y_d with the exponential decay rate $e^{-\delta t}$.

Example.

Let us consider the system

$$(3.19)\quad \begin{cases} \dfrac{\partial u(t,x)}{\partial t}=\dfrac{\partial^2 u(t,x)}{\partial x^2}+4\pi^2 u(t,x), & x\varepsilon(0,0.2)U(0.2,0.7)U(0.7,1) \\ u(t,0)=u(t,1)=0 \quad t>0, \; u(0,x)=u_0(x) \\ [u_x(t,0.5)]_{0.5-}^{0.5+}=d \end{cases}$$

where $[f(s)]_{s-}^{s+}$ denotes the change of the value of the function at the point s and d is an unknown constant.

The control is given by

$$(3.20)\quad \begin{cases} [u_x(t,0.2)]_{0.2-}^{0.2+}=f_1(t), & [u(t,0.2)]_{0.2-}^{0.2+}=0 \\ [u_x(t,0.7)]_{0.7-}^{0.7+}=f_2(t), & [u(t,0.7)]_{0.7-}^{0.7+}=0 \end{cases} \quad t>0.$$

The mesurements at the points 0.3 and 0.6 should be regulated so that

$$(3.21)\quad \begin{cases} y_1(t)=u(t,0.3) \rightarrow 1=y_{d1} \\ y_2(t)=u(t,0.6) \rightarrow 3=y_{d2}. \end{cases}$$

For this example we consider the case when $z_1(t)=y_1(t)$, $z_2(t)=y_2(t)$.

The operator $A:D(A) \rightarrow L^2(0,1)$, $Au=u''+4\pi^2 u$, where $D(A)=\{u\varepsilon L^2(0,1)\,|\,u,u'$ are absolutely continuous, $u''\varepsilon L^2(0,1)$, $u(0)=u(1)=0\}$, has the eigenset

$$\phi_n(x)=\sqrt{2}\sin n\pi x, \quad \lambda_n=-(n\pi)^2+4\pi^2, \quad n=1,2,---.$$

Now we may define a set of Hilbert spaces

$$H_t=\{u=\sum_{n=1}^{\infty}(u,\phi_n)_0\phi_n \,|\, \sum_{n=1}^{\infty}|\beta_n|^{2t}|(u,\phi_n)_0|^2<\infty\}$$

with the following inner product

$$(u,v)_t=\sum_{n=1}^{\infty}|\beta_n|^{2t}(u,\phi_n)_0(v,\phi_n)_0$$

where $H_0=L^2(0,1)$ and $\beta_n=\lambda_n-4\pi^2$, $n=1,2,---$.

We note that $A\varepsilon L(H_t,H_{t-1})$ is a closed linear operator $A:D(A)=H_t \rightarrow H_{t-1}$ with the same eigenfunctions for all $t\varepsilon R$.

It can be shown that the problem (3.19),(3.20),(3.21) can be written as the control problem

$$(3.22)\quad \begin{cases} \dfrac{du}{dt}=Au(t)+Bf(t)+w \\ y(t)=Cu(t) \end{cases}$$

in the Hilbert space $X=H_{-1/2}$. The operators A, B ,C, M and the disturbance w are given as

$$Au=\sum_{n=1}^{\infty}\lambda_n(u,\phi_n)_0\phi_n \qquad \text{for all } u\varepsilon H_{1/2}$$

$$Bf(t)=-[\sum_{n=1}^{\infty}\phi_n(0.2)\phi_n,\ \sum_{n=1}^{\infty}\phi_n(0.7)\phi_n]\begin{bmatrix} f_1(t) \\ f_2(t)\end{bmatrix}$$

$$w=-d\sum_{n=1}^{\infty}\phi_n(0.5)\phi_n$$

$$Cu=Mu=\begin{bmatrix}\sum_{n=1}^{\infty}(u,\phi_n)_0\phi_n(0.3) \\ \sum_{n=1}^{\infty}(u,\phi_n)_0\phi_n(0.6)\end{bmatrix}=\begin{bmatrix}u(0.3) \\ u(0.6)\end{bmatrix} \qquad \text{for all } u\varepsilon H_{1/2}.$$

The operator B is a bounded linear operator from E^2 to $H_{-1/2}$. Moreover since all $u\varepsilon H_{1/2}\subset C(0,1)$, a pointwise observation at $x_o\varepsilon(0,1)$ can be defined with the aid of an element $c=\sum_{n=1}^{\infty}\phi_n(x_o)\phi_n\varepsilon H_{-1/2}$. Then

$$\left|\sum_{n=1}^{\infty}(u,\phi_n)_0\phi_n(x_o)\right|\leqq \sum_{n=1}^{\infty}\frac{1}{|\beta_n|}\phi_n^2(x_o)\ \sum_{n=1}^{\infty}|\beta_n|(u,\phi_n)_0^2$$

$$=||c||_{-1/2}||u||_{1/2}=||c||_{-1/2}||(A-4\pi^2)u||_{-1/2}$$

$$\leqq 4\pi^2||c||_{-1/2}||u||_{-1/2}+||c||_{-1/2}||Au||_{-1/2} \qquad \text{for all } u\varepsilon H_{1/2}=D(A)$$

which proved that the pointwise observation are A-bounded. This implies that $C:H_{1/2}\to E^2$ is A-bounded also. Thus the presented theory can be applied.

Now $\lambda_2=0$, $\lambda_3=-5\pi^2$ and then we can take $N\geqq 2$. Here choose

$$PX=\text{span}\{\phi_n(x);n=1,2\},\quad QX=\text{span}\{\phi_n(x);n=3,4,---\},$$

then N=2 and

$$U_Q(t)u_0=\sum_{n=3}^{\infty}\exp(\lambda_n t)\phi_n(u_0,\phi_n)_0.$$

In this case

$$(3.23)\qquad ||U_Q(t)||\leqq\exp(\lambda_3 t)=e^{-5\pi^2 t}$$

which implies that K=1 and $\sigma=5\pi^2$ in (1.8).

Relative to the basis ϕ_1,ϕ_2 for PX, we have

$$A_P=\begin{bmatrix}3\pi^2 & \\ & 0\end{bmatrix},\quad PB=-\begin{bmatrix}\phi_1(0.2) & \phi_1(0.7)\\ \phi_2(0.2) & \phi_2(0.7)\end{bmatrix},\quad C_P=M_P=\begin{bmatrix}\phi_1(0.3) & \phi_2(0.3)\\ \phi_1(0.6) & \phi_2(0.6)\end{bmatrix}$$

$$Pw=-d\begin{bmatrix}\phi_1(0.5)\\ \phi_2(0.5)\end{bmatrix}.$$

Moreover the operator $S\varepsilon L(X,E^2)$ is defined by

$$Sw=S_QQw=-C_QA_Q^{-1}Qw=-\sum_{n=3}^{\infty}\frac{1}{\lambda_n}w_n\begin{bmatrix}\phi_n(0.3)\\ \phi_n(0.6)\end{bmatrix},\quad w\varepsilon H_{-1/2}$$

where $w_n=(w,\phi_n)_0$. Then

$$SB=-C_QA_Q^{-1}QB=\begin{bmatrix}\sum_{n=3}^{\infty}\frac{1}{\lambda_n}\phi_n(0.2)\phi_n(0.3) & \sum_{n=3}^{\infty}\frac{1}{\lambda_n}\phi_n(0.7)\phi_n(0.3)\\ \sum_{n=3}^{\infty}\frac{1}{\lambda_n}\phi_n(0.2)\phi_n(0.6) & \sum_{n=3}^{\infty}\frac{1}{\lambda_n}\phi_n(0.7)\phi_n(0.6)\end{bmatrix}.$$

Next we investigate controllability of the 4 dimensional system (2.11) by Remark 1. Since $\phi_n(0.2)\neq 0$, $\phi_n(0.7)\neq 0$, n=1,2, rank(PB A_PPB]=2. Thus if the condition

$$\text{rank}\begin{bmatrix}SB & C_P\\ PB & A_P\end{bmatrix}=4$$

holds, the system (2.11) is controllable.

Analogously sufficient conditions for observability of the 2 dimensional system (A_P,M_P) are $\phi_n(0.3)\neq 0$, n=1,2 (or $\phi_n(0.6)\neq 0$, n=1,2).

So an output feedback regulator through an identity observer for our systen is given by

$$\begin{bmatrix}f_1(t)\\ f_2(t)\end{bmatrix}=DSv(t)+F_0Pv(t)+D\int_0^t(y(s)-y_d)ds \tag{3.24}$$

$$\begin{cases}\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}+4\pi^2 v-G\begin{bmatrix}v(t,0.3)\\ v(t,0.6)\end{bmatrix}+G\begin{bmatrix}u(t,0.3)\\ u(t,0.6)\end{bmatrix}\\ v(t,0)=v(t,1)=0,\quad v(0,x)=0\\ [v_x(t,0.2)]_{0.2-}^{0.2+}=f_1(t),\quad [v(t,0.2)]_{0.2-}^{0.2+}=0\\ [v_x(t,0.7)]_{0.7-}^{0.7+}=f_2(t),\quad [v(t,0.7)]_{0.7-}^{0.7+}=0\end{cases} \tag{3.25}$$

where

$$D=\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \qquad F_0Pv(t)=\begin{bmatrix} s_{11}v_1(t)+s_{12}v_2(t) \\ s_{21}v_1(t)+s_{22}v_2(t) \end{bmatrix}$$

$$G\begin{bmatrix} v(t,0.3) \\ v(t,0.6) \end{bmatrix}=\sum_{n=1}^{2} g_{1n}v(t,0.3)\phi_n(x)+\sum_{n=1}^{2} g_{2n}v(t,0.6)\phi_n(x)$$

$$F_0=\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}, \qquad G_0=\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^T.$$

We construct the matrices D, F_0 and G_0 such that all the eigenvalues of

$$A_{fP}=\begin{bmatrix} 0 & C_P \\ 0 & A_P \end{bmatrix}+\begin{bmatrix} SB \\ PB \end{bmatrix}[D \quad F_0]$$

and $A_{GP}=A_P-G_0M_P$ have negative real parts.

Moreover, relative to the basis $\phi_1,---,\phi_L$ for P_LX, the L dimensional observer (3.18) becomes

$$(3.26) \quad \begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \vdots \\ \dot{v}_L(t) \end{bmatrix}=\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_L \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_L(t) \end{bmatrix}-\begin{bmatrix} \phi_1(0.2) & \phi_1(0.7) \\ \phi_2(0.2) & \phi_2(0.7) \\ \vdots & \vdots \\ \phi_L(0.2) & \phi_L(0.7) \end{bmatrix}\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

$$+\begin{bmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \\ & 0 \end{bmatrix}\begin{bmatrix} u(t,0.3) \\ u(t,0.6) \end{bmatrix}-\begin{bmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \\ & 0 \end{bmatrix}\begin{bmatrix} \phi_1(0.3) & \phi_2(0.3)---\phi_L(0.3) \\ \phi_1(0.6) & \phi_2(0.6)---\phi_L(0.6) \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_L(t) \end{bmatrix}$$

and

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}=-D\sum_{n=3}^{L}\frac{1}{\lambda_n}v_n(t)\begin{bmatrix} \phi_n(0.3) \\ \phi_n(0.6) \end{bmatrix}+F_0Pv(t)+D\int_0^t(y(s)-y_d)ds.$$

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