数理解析研究所講究録
491巻 1983年 107-113

Some martingale identities and inequalities

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0. Summary. We consider moment type inequalities and identities for continuous time martingales. An application is given for processes with independent increments.

1. General denotations. We assume that (Ω,F,P) is a complete probability space with a family (F_t), $t \geq 0$, of σ-algebras satistying the standart conditions (nondecreasing, right continuous etc, [1], [2]; the paper [2] contains a short and good survey of the theory of stochastic integration). Any random process will be supposed to be F_t-adapted and "cadlag" (that is right continuous and having limits on the left). Denote by M $(M_{loc}, M^c_{loc}, M^d_{loc}, M^2_{loc})$ classes of uniformly integrable martingales (local, continuous local, pure discontinuous local, local square-integrable, respectively) with respect to the family (F_t).

We denote by $p = p(\omega,dt,dx)$ the integral random measure of jumps $\Delta X_s = X_s - X_{s-}$ of some given (model) process X_s,

$$p(\omega, (0,t], \Gamma) = \sum_{0<s\leq t} I(\Delta X_s \in \Gamma),\ \Gamma \in B(R^m\setminus\{0\}).$$

Here $B(R^m\setminus\{0\})$ is a symbol of the Borel algebras of sets from $R^m\setminus\{0\}$. Also, let $q = q(\omega,dt,dx)$ be a compensator, or the dual predictable projection to the measure p ([1], [2]). Stochastic integrals of predictable functions $f = f(\omega,s,x)$ with respect to the measures $q(\omega,ds,dx)$ and $p(\omega,ds,dx)-q(\omega,ds,dx)$ will be denoted by symbols

$$\int_0^t \int_{R^m\setminus\{0\}} f(\omega,s,x)\circ q(\omega,ds,dx)$$

and

$$\int_0^t \int_{R^m\setminus\{0\}} f(\omega,s,x)\circ (p(\omega,ds,dx)-q(\omega,ds,dx))$$

respectively. In what follows we omit the variables (ω,s,x) and index $R^m\setminus\{0\}$. These stochastic integrals are defined for proper classes of (vector-valued) functions f ([1], [2]). In particular,

$$\int_0^\infty \int \frac{|f|^2}{1+|f|^2}\circ q < \infty \quad \text{a.s.} \Longrightarrow \int_0^t \int f\circ (p-q) \in M^d_{loc}.$$

If $\mu_t \in M^2_{loc}$ there exists a unique predictable process $\langle\mu\rangle_t$ such that $(|\mu_t|^2 - \langle\mu\rangle_t) \in M_{loc}$. Note that

$$\int_0^\infty \int |f|^2\circ q < \infty \quad \text{a.s.} \Longrightarrow \int_0^t \int f\circ (p-q) \in M^2_{loc},$$

and moreover, if the compensator q is continuous that is

$$q(\omega,\{t\},\Gamma) = 0 \quad \text{for any } \Gamma \in B(R^m\setminus\{0\}) \tag{1}$$

then

$$\left\langle \int_0^t \int f\circ (p-q)\right\rangle = \int_0^t \int |f|^2\circ q.$$

The letter C will denote any positive constant. We shall use the usual denotations for a maximal functions,

$$\mu^* = \sup_{t\geq 0} |\mu_t|.$$

2. Structure of martingales under consideration. It is well-known that any process $\mu_t \in M_{loc}$ can be represented as a sum,

$$\mu_t = \mu^c_t + \mu^d_t, \quad (\mu^c_t \in M^c_{loc},\ \mu^d_t \in M^d_{loc}).$$

In what follows we suppose that $\mu_0=0$ and

$$\mu_t^d = \int_0^t \int f\circ(p-q)$$

with a proper function f (predictable, $\int_0^t \int |f|^2(1+|f|^2)^{-1}\circ q < \infty$ a.s.). Note that if the model process X_t is a local martingale then ([1])

$$X_t = X_t^c + \int_0^t \int x\circ(p-q), \quad (X_t^c \in M_{loc}^c).$$

We suppose for simplity of formulations that the compensator q is *continuous*. In this case we have a simple formula for the quadratic characteristic of a martingale $\mu_t \in M_{loc}^2$

$$\langle\mu\rangle_t = \langle\mu^c\rangle_t + \int_0^t \int |f|^2\circ q.$$

3. Moment inequalities. In what follows we denote by $\phi(x)$, $x \geq 0$, a nondecreasing continuous function such that

$$\phi(2x) \leq C\phi(x) \quad \text{for all } x \geq 0 \text{ and } \phi(0) = 0.$$

(F.e. $\phi(x) = x^p L(x)$, $0\leq p<\infty$, with $L(x)$ being a slowly varying function).

Theorem 1. *Let* $\mu_t = \mu_t^c + \int_0^t \int f\circ(p-q)$, $\mu_t^c \in M_{loc}^c$, *and the compensator* q *satisfy the condition* (1). *If* ϕ *is a concave function then for any* $\alpha \in [1,2]$

$$E\phi(\mu^{*\alpha}) \leq CE\phi(\langle\mu^c\rangle_\infty^{\alpha/2}) + CE\phi\left(\int_0^\infty \int |f|^\alpha\circ q\right)$$

where constants C *depend only on* ϕ *and* α.

If ϕ *is a convex function then*

$$CE\phi(\langle\mu\rangle_\infty^{1/2}) + CE\left(\int_0^\infty \int \phi(|f|)\circ q\right) \leq E\phi(\mu^*) \leq CE\phi(\langle\mu\rangle_\infty^{1/2}) +$$

$$CE\left(\int_0^\infty \int \phi(|f|)\circ q\right),$$

where constants C depend only on ϕ.

Remarks. In case of $\phi(x) = x^p$ and an absolute continuous compensator q these inequalities were proved in [3]. In case of discrete time martingales similar inequalities can be found in [4], [5].

4. Moment identities. Here we consider one-dimentional martingales μ_t which have the representation mentioned above with a continuous compensator q.

Define polynomials $V_n(y_1) = V_n(y_1,\cdots,y_n)$ by help of the next recurrent formulas

$$V_0(y_1) = 1,\ V_1(y_1) = 1,$$

$$V_{n+1}(y_1) = y_1 V_n(y_1) - \sum_{j=0}^{n-1} \binom{n}{j} y_{n+1-j} V_j(y_1), \quad (n=2,3,\cdots).$$

Note if $y_i=0$ for $i \geq 3$ then

$$V_n(y_1) = y_2^{n/2} He_n\left(\frac{y_1}{\sqrt{y_2}}\right), \quad n = 1,2,\cdots,$$

where $He_n(x) = (-1)^n \exp(\frac{1}{2}x^2)\frac{d^n}{dx^n}\exp(-\frac{1}{2}x^2)$ are Hermitian polynomials. We shall use the following denotations

$$\bar{V}_n(\mu_t) = V_n\left(\mu_t+C_1, \langle\mu\rangle_t+C_2, \int_0^t\int f^3\circ q+C_3, \cdots, \int_0^t\int f^n\circ q+C_n\right)$$

where C_i are some constants, and $\bar{V}_n(0) = V_n(C_1, C_2, \ldots, C_n)$.

Theorem 2. Let $\mu_t = \mu_t^c + \int_0^t\int f\circ(p-q)$, $\mu_t^c \in M_{loc}^c$ and the compensator q satisfy the condition (1). If

$$E(\langle\mu^c\rangle_\infty^{1/2}) + E\left(\int_0^\infty\int |f|^\alpha\circ q\right)^{1/2} < \infty$$

for some $\alpha \in [1,2]$, n=1, or

$$E(\langle\mu\rangle_\infty)^{n/2} + E(\int_0^\infty \int |f|^n \circ q) < \infty$$

for $n=2,3,\cdots$, then

$$E\bar{V}_k(\mu_\infty) = \bar{V}_k(0) \quad (k=1,\cdots n) . \qquad (2)$$

Remarks. In case of an absolute continuous compensator q theorem 2 was proved in [3] (in a little different form).

The conditions of theorem 2 guarantee also that $V_n(\mu_t) \in M$ and moreover, that $E \sup_{t \geq 0} |V_n(\mu_t)| < \infty$.

5. An application. The moment identities (2) may be used, for example, for calculating moments of first passage time for processes with independent increments through moving boundaries. Some examples for the case when μ_t is a standart wiener process can be found in [6], and in the recent paper Farebee [7]. Here is an other example.

Let X_t be a stochastically continuous process with independent increments having only positive jumps (that is its spectral measure $Q(dx)$ equals zero for $x < 0$). Suppose $EX_1=0$, $EX_1^2=1$ (if it exists), and consider a stopping time

$$\sigma_a = \inf\{t \geq 0: \; X_t \leq at^{1/2}-b\}, \quad (a>0, \; b>0).$$

Then under the condition $E(X_1^+)^n < \infty$, $(n=1,2,\cdots)$

$$E\sigma_a^{k/2} < \infty \iff a > \bar{z}_k, \quad k=1,2,\cdots n$$

where $\bar{z}_k = \max(z : He_k(z)=0)$.

The moments $E\sigma_a^{k/2}$ can be calculated by help of identities (2). For example,

$$E\sigma_a^{1/2} = \frac{b}{a} \quad (a > 0), \quad E\sigma_a = \frac{b^2}{a^2-1} \quad (a > 1), \cdots .$$

6. The Wald identity for continuous martingale. In case of $\mu_t \in M^c_{loc}$ we can slightly weaken the condition of the theorem 2 (for $n = 1$).

Denote by N a class of nonnegative continuous nondecreasing functions such that

$$\int_1^\infty \frac{f(t)}{t^{3/2}}\, dt = \infty.$$

(F.e. $f(t) = \sqrt{t}\,(\log(t+1))^{-1} \in N$ and so on).

Theorem 3. Let $\mu_t \in M^c_{loc}$, $\langle\mu\rangle_\infty < \infty$ a.s. and $E|\mu_\infty| < \infty$. If $f \in N$ then

$$Ef(\langle\mu\rangle_\infty) < \infty \Longrightarrow E\mu_\infty = 0.$$

Remarks. If $P(\langle\mu\rangle_\infty > t)\sqrt{t} = o(1)$, $t \to \infty$, then there exists a function $f \in N$ such that $Ef(\langle\mu\rangle_\infty) < \infty$. This fact and the theorem [3] (in a little different form) were mentioned by the author in [8]. In the paper [9] it was shown that under the additional condition $\sup_{t\ge 0} E|\mu_t| < \infty$

$$\lim_{t\to\infty} P(\langle\mu\rangle_\infty > t)\sqrt{t} = 0 \Longleftrightarrow \lim_{t\to\infty} E|\mu_\infty - \mu_t| = 0.$$

Our approach is based on some simple facts about first passage times for a wiener process and differs from [9] (see details of proofs in [10]).

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