Classification of Nash manifolds

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1. Introduction

In this paper we show when two Nash manifolds are Nash diffeomorphic. A semi-algebraic set in a Euclidean space is called a Nash manifold if it is an analytic manifold, and an analytic function on a Nash manifold is called a Nash function if the graph is semi-algebraic. We define similarly a Nash mapping, a Nash diffeomorphism, a Nash manifold with boundary, etc. It is natural to ask a question whether any two $C^\infty$ diffeomorphic Nash manifolds are Nash diffeomorphic. The answer is negative. We give a counter-example in Section 5. The reason is that Nash manifolds determine uniquely their "boundary". In consideration of the boundaries, we can classify Nash manifolds by Nash diffeomorphisms as follows. Let $M, M_1, M_2$ denote Nash manifolds.

Theorem 1. There exist a compact real non-singular affine algebraic set $X$, a non-singular algebraic subset $Y$ of $X$ of codimension 1, and a union $M'$ of connected components of $X-Y$ such that $M$ is Nash diffeomorphic to $M'$ and that the closure $\bar{M}'$ of $M'$ is a Nash manifold with boundary $Y$. Here $Y$ is empty if $M$ is compact.
In the above we call $\tilde{M}$ a compactification of $M$.

Theorem 2. Let $N_1, N_2$ be any respective compactifications of $M_1, M_2$. Then the following are equivalent.

(i) $M_1$ and $M_2$ are Nash diffeomorphic.
(ii) $N_1$ and $N_2$ are Nash diffeomorphic.
(iii) $N_1$ and $N_2$ are $C^\infty$ diffeomorphic.

By the h-cobordism theorem [5] we have

Corollary 3. Assume that $M_1$ and $M_2$ are $C^\infty$ diffeomorphic, that the dimension of $M_1$ is not 3, 4 nor 5, and that if $\dim M_1 \geq 6$, for any compact subset $A$ of $M_1$ there exists a compact subset $A' \supset A$ of $M_1$ such that $M_1 - A'$ is simply connected. Then $M_1$ and $M_2$ are Nash diffeomorphic.

The correspondence $M \to$ the compactification of $M$ shows the following.

Corollary 4. The Nash diffeomorphic classes of all Nash manifolds are in (1-1)-correspondence with the $C^\infty$ diffeomorphic classes of all $C^\infty$ compact manifolds with or without boundary.

The next corollaries may be useful when we consider Nash manifolds and Nash functions.

Corollary 5. Let $M_1 \supset M'_1, M_2$ be Nash manifolds and a compact Nash submanifold. Let $f: M_1 \to M_2$ be a $C^\infty$ mapping such that
$f|_{M_1}$ is a Nash mapping. Then we can approximate $f$ by Nash mappings fixing on $M_1$ in the compact-open $C^\infty$ topology.

Corollary 6. Assume that $M$ is compact and contained in $\mathbb{R}^n$. Then there exist Nash functions $f_1, \ldots, f_p$ on $\mathbb{R}^n$ such that the common zero points set of $f_1, \ldots, f_p$ is $M$ and that grad $f_1, \ldots, \text{grad } f_p$ on $M$ span the normal bundle of $M$ in $\mathbb{R}^n$.

2. Preparation

See [3] for the fundamental properties of semi-algebraic sets.

Lemma 7. Let $M \subset \mathbb{R}^n$ be a Nash manifold. Then there exists a Nash tubular neighborhood $U$ of $M$ in $\mathbb{R}^n$, (i.e. $U$ is a Nash manifold and the orthogonal projection $p: U \to M$ is a Nash mapping).

Proof. Let $\overline{M}$ be the Zariski closure of $M$ in $\mathbb{R}^n$. Let $\text{Sing}(\overline{M})$ denote the set of singular points of $\overline{M}$. Then $M - \text{Sing}(\overline{M})$ is open and dense in $M$. Consider the normal bundle

$$N = \{(x, y) \in M \times \mathbb{R}^n | y \text{ is a normal vector of } M \text{ at } x \text{ in } \mathbb{R}^n\}.$$ 

Then clearly $N$ is an analytic manifold. Moreover $N$ is semi-algebraic. The reason is the following. We define the normal bundle $\tilde{N}$ of $\overline{M} - \text{Sing}(\overline{M})$ in the same way. Since $\tilde{N}$ is an algebraic subset of $(\overline{M} - \text{Sing}(\overline{M})) \times \mathbb{R}^n$, $\tilde{N} \cap (M \times \mathbb{R}^n)$ is semi-algebraic. The equality
\[ \tilde{N} \cap (M \times \mathbb{R}^n) = N \cap ((M - \text{Sing}(\tilde{M})) \times \mathbb{R}^n) \]

and the dense property of \( M - \text{Sing}(\tilde{M}) \) in \( M \) imply that \( N \) is the topological closure of \( \tilde{N} \cap (M \times \mathbb{R}^n) \) in \( M \times \mathbb{R}^n \). Hence \( N \) is semi-algebraic.

The mapping \( q : N \ni (x, y) \mapsto x + y e \mathbb{R}^n \) is obviously of Nash class. Let \( E_1 \) be the set of critical points of the mapping \( q \circ q : N \times N \to \mathbb{R}^n \times \mathbb{R}^n \). Then \( N \times N - E_1 \) contains

\[ \Delta_1 = \{(z_1, z_2) \in N \times N | z_1 = z_2 = (x, 0)\} \]

Let \( E_2 \) be the set of all points \((z_1, z_2) \in N \times N\) such that \( q(z_1) = q(z_2) \). Then \( E_2 \) is a closed semi-algebraic subset of \( N \times N \) and contains the diagonal

\[ \Delta_2 = \{(z_1, z_2) \in N \times N | z_1 = z_2\} \]

Moreover the topological closure \( \overline{E_2 - \Delta_2} \) does not intersect with \( \Delta_1 \) because of the existence of \( C^\infty \) tubular neighborhoods of \( M \). Hence \( E_1 \cup \overline{E_2 - \Delta_2} \) is a closed semi-algebraic subset of \( N \times N \) which does not intersect with \( \Delta_1 \).

Let \( \varphi \) be a positive continuous function on \( M \) defined by

\[ \varphi(x) = \text{dist}((x, 0, x, 0), E_1 \cup \overline{E_2 - \Delta_2}) \]

It is easy to see that any distance function from a semi-algebraic set is semi-algebraic (i.e. the graph is semi-algebraic). Hence \( \varphi \) is semi-algebraic. Put
\[ N' = \{ (x,y) \in \mathbb{N} | 2|y| < \Phi(x) \}. \]

Then \( N' \) is an open semi-algebraic subset of \( \mathbb{N} \). We want to see that the restriction of \( q \) to \( N' \) is a Nash diffeomorphism into \( \mathbb{R}^n \). It is trivial that the restriction is an immersion. Assume the existence of points \( z_1 = (x_1, y_1) \) and \( z_2 = (x_2, y_2) \) in \( N' \) such that \( q(z_1) = q(z_2) \), \( z_1 \neq z_2 \). Then we have

\[
\begin{align*}
x_1 + y_1 & = x_2 + y_2, \\
\text{dist}^2((x_1,0,x_1,0),(z_1,z_2)) & = |x_1 - x_2|^2 + y_1^2 + y_2^2 \\
\text{dist}^2((x_2,0,x_2,0),(z_1,z_2)) & \geq \Phi(x_1)^2, \Phi(x_2)^2,
\end{align*}
\]

and

\[ 2 |y_1| < \Phi(x_1), \ 2 |y_2| < \Phi(x_2). \]

It follows that \( |x_1 - x_2|^2 = |y_1 - y_2|^2 \) and

\[ |x_1 - x_2|^2 + y_1^2 + y_2^2 > 4y_1^2, \ 4y_2^2. \]

Hence \( |y_1 - y_2|^2 > y_1^2 + y_2^2 \). This is a contradiction. Therefore \( q(N') \) is a Nash tubular neighborhood of \( M \) in \( \mathbb{R}^n \). The proof is complete.

The following lemma will be used in the proof of Theorem 2, but this may be interesting itself. The case of polynomials on a Euclidean space was treated in Remark 6 in [11].

**Lemma 8.** Let \( M \subset \mathbb{R}^n \) be a Nash manifold closed in \( \mathbb{R}^n \). Let \( f_1, f_2 \) be positive proper Nash functions on \( M \). Then there exists a \( C^\infty \) diffeomorphism \( \tau \) of \( M \) such that \( f_1 \circ \tau \) and \( f_2 \) are equal outside a bounded subset of \( M \).
Proof. The case where \( M \) is compact is trivial. Hence we assume \( M \) to be not compact. Let \( \tilde{r}_1, \tilde{r}_2 \) be the extension of \( r_1, r_2 \) respectively onto a Nash tubular neighborhood \( U \) of \( M \) defined by \( \tilde{r}_i = r_i \circ p, i=1,2 \), where \( p \) is the orthogonal projection. Then \( \tilde{r}_i \) are Nash functions, since any composition of Nash mappings is of Nash class. We regard \( \text{grad} \tilde{r}_i, i=1,2 \) as Nash mappings from \( U \) to \( \mathbb{R}^n \) also. The restrictions of \( \text{grad} \tilde{r}_1 \) and \( \text{grad} \tilde{r}_2 \) to \( M \) are vector fields of \( M \). Let the restrictions be denoted by \( w_1, w_2 \) respectively. Put

\[
B = \{ x \in M | \langle w_1, w_2 \rangle = -|w_1||w_2| \}.
\]

Here \( \langle , \rangle \) means the inner product as vectors. Then \( B \) is semi-algebraic because of

\[
B = M \cap \{ x \in U | \langle \text{grad} \tilde{r}_1(x), \text{grad} \tilde{r}_2(x) \rangle = -|\text{grad} \tilde{r}_1(x)||\text{grad} \tilde{r}_2(x)| \}.
\]

Obviously \( B \) is the set of points \( x \) where \( w_1 \) is zero or \( w_2 \) is a multiple of \(-w_1 \) and a real non-negative number.

We will prove by reduction to absurdity that \( B \) is bounded. Assume it to be unbounded. As \( \mathbb{R}^n \) is Nash diffeomorphic to \( S^n \backslash \{ \text{a point } a \} \) by the stereographic projection, we identify them. The germ of \( B \) at \( a \) is not empty. Hence, considering the germ, we obtain easily an unbounded one-dimensional semi-algebraic set \( B' \subset B \) (see [3]). We can assume that \( B' \) is a Nash manifold with boundary and Nash diffeomorphic to \([0, \infty)\), because the set of singular points of one-dimensional semi-algebraic set is a semi-algebraic set of dimension 0. Let \( v \) be a \( C^\infty \) non-singular

\[
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\]
vector field on $B'$. Then, by the definition of $B$, we have

$$\text{vf}_1(x) \times \text{vf}_2(x) \leq 0 \quad \text{for} \quad x \in B'.$$

On the other hand, any non-constant Nash function defined on $[0, \infty)$ is monotone outside a bounded subset, because the set of critical points is a semi-algebraic set of dimension 0. Hence one of the functions $f_1|_{B'}$ and $f_2|_{B'}$ is monotone decreasing outside a bounded subset. This contradicts the fact that $f_1, f_2$ are proper and positive.

Let $K$ be a large real number, let $\varphi$ be a $C^\infty$ function on $M$ such that

$$0 \leq \varphi \leq 1, \quad \varphi = \begin{cases} 0 & \text{for} \quad |x| \leq K^{1/2} \\ 1 & \text{for} \quad |x| \geq (2K)^{1/2}. \end{cases}$$

Put

$$L = M \cap \{ |x| = K^{1/2} \}, \quad L' = M \cap \{ |x| \geq (2K)^{1/2} \}, \quad L'' = M \cap \{ |x| \leq K^{1/2} \}.$$

For any real $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$, the vector field $w' = c_1 w_1 + c_2 w_2$ is non-singular outside $B$ and satisfies $w'|f_1$, $w'|f_2 > 0$ at any point $x \notin B$ such that $c_1 |w_{1x}| = c_2 |w_{2x}|$.

Choose $K$ so that $L' \cap B = \emptyset$. Put

$$w = \frac{w_1}{|w_1|} + \varphi \frac{w_2}{|w_2|} \quad \text{on} \quad L'.$$

Then $w, w_1$ and $w_2$ are non-singular vector fields on $L'$. Moreover $w|f_1, w|f_2$ are positive on $L', L''$ respectively. It is sufficient to consider the case

$$f_1(x) = x_1^2 + \ldots + x_n^2 \quad \text{for} \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$
Since $L$ is a level of $f^1$, $L$ is smooth, and $w^1$ is transversal to $L$.

On any maximal integral curve of $w$, $f^1$ is non-singular and monotone, and the set of values is $[K, \infty)$. Let $\psi_t$ be the local parameter group of transformations of $L'$ defined by $w$. Then $\psi_t$ is well-defined for $0 \leq t < \infty$. Put

$$\pi'(z,t) = \psi_t(z) \quad \text{for} \quad (z,t) \in L \times [0, \infty).$$

It follows that $\pi'$ is a diffeomorphism onto $L'$. The mapping

$$(z,t) \mapsto (z, f^1 \circ \pi'(z,t) - K)$$

is a diffeomorphism of $L \times [0, \infty)$. Let $(z,t) \mapsto (z, s(z,t))$ be the inverse diffeomorphism. Put

$$\pi(z,t) = \pi'(z, s(z,t)) \quad \text{for} \quad (z,t) \in L \times [0, \infty).$$

Then $\pi$ is a diffeomorphism from $L \times [0, \infty)$ to $L'$ such that

$$f^1 \circ \pi(z,t) = t + K \quad \text{for} \quad (z,t) \in L \times [0, \infty).$$

By the definition of $\pi$ and $\pi'$ we have a positive $C^\infty$ function $\rho$ on $L'$ such that $\pi_\# \left( \frac{\partial}{\partial t} \right) = \rho w$.

It follows from $\pi(L \times \{t > K\}) = L''$ that

$$\frac{\partial f^2 \circ \pi}{\partial t}(z,t) > 0 \quad \text{for} \quad t \geq K.$$

Hence, for each $x \in L$, the t-function $f^2 \circ \pi(x,t)$ on $[K, \infty)$ is proper and non-singular. Choose real $K' (> K)$ so that
\[ f_2 \circ \pi(x,t) > K \quad \text{for} \ (x,t) \in L \times [K', \infty). \]

Then we have a \( C^\infty \) function \( f_3 \) on \( L \times [0, \infty) \) such that \( f_3(x,t) \)
\( 0 \leq t < \infty \), is \( C^\infty \) regular for each fixed \( x \in L \), that \( f_3(x,t) = t + K \)
in a neighborhood of \( L \times 0 \) and that \( f_3(x,t) = f_2 \circ \pi(x,t) \) for
\( (x,t) \in L \times [K', \infty) \). It follows that \( (x,t) \rightarrow (x, f_3(x,t) - K) \) is a
diffeomorphism of \( L \times [0, \infty) \). Let \( \pi'' : (x,t) \rightarrow (x, s'(x,t)) \) be
the inverse. Then we see that

\[ f_2 \circ \pi \circ \pi''(x,t) = t + K \quad \text{if} \ s'(x,t) \geq K'. \]

Hence

\[ f_1 \circ \pi = f_2 \circ \pi \circ \pi'' \quad \text{if} \ s'(x,t) \geq K'. \]

Since \( s'(x,t) = t \) in a neighborhood of \( L \times 0 \), we can extend
\( \pi \circ \pi''^{-1} \circ \pi^{-1} \) onto \( M \) so that the extension \( \bar{\pi} \) is the identity
on \( M - L' \). Then \( f_1 \circ \bar{\pi} = f_2 \) outside a bounded set. Hence Lemma
is proved.

3. Proofs of Theorems 1,2

For the sake of brevity we assume that \( M, M_1 \) and \( M_2 \) are
connected. We also assume that the manifolds are not compact,
because the other case is well-known. Let \( n' \) be the dimension
of the manifolds. Let \( G_{m,m'} \) denote the Grassmann manifold
of \( m \)-linear subspaces in \( \mathbb{R}^{m+m'} \). Put

\[ E_{m,m'} = \{ (\lambda,x) \in G_{m,m'} \times \mathbb{R}^{m+m'} \mid x \in \lambda \}. \]
Then $G_{m,m'}$ has naturally affine non-singular algebraic structure [7].

Let $\mathfrak{M}'$ denote $\mathfrak{M}' - M'$ if $M'$ is a manifold contained in $\mathbb{R}^n$ and the usual boundary if $M'$ is a compact manifold with boundary.

**Proof of Theorem 1.** (1) First we reduce the problem to the case in which there exist a real compact non-singular algebraic set $X \subseteq \mathbb{R}^n$ and an algebraic subset $Z$ of $X$ satisfying the following conditions, (this was shown in the proof of Proposition 1 in [9]).

(i) $M$ is a connected component of $X - Z$.

(ii) For every point $a \in Z$, there exists a smooth rational mapping $\xi$ from $X$ to $\mathbb{R}^{n''}$ for some integer $n'' \leq n'$ such that $\xi(a) = 0$, that

$$
Z \left\{ \begin{array}{ll}
\subseteq & \xi^{-1}(\{(x_1, \ldots, x_{n''}) \in \mathbb{R}^{n''} | x_1 \ldots x_{n''} = 0\}) \\
on U & \text{on } U \\
on X & \text{on } X
\end{array} \right.
$$

where $U$ is a neighborhood of $a$ in $X$, and that $\xi$ is a submersion on $U$. In this case we say that $Z$ has only normal crossings at $a$ in $X$.

**Proof.** The boundary $\mathfrak{M}'$ is a closed semi-algebraic set in $\mathbb{R}^n$. By Lemma 6 in [6], there exists a continuous function $\eta$ on $\mathbb{R}^n$ such that $\eta^{-1}(0) = \mathfrak{M}'$ and that the restriction of $\eta$ to $\mathbb{R}^n - \mathfrak{M}'$ is of Nash class, (see the remark after Proposition 1 in [9]). Consider the graph of the restriction of $1/\eta$ to $M$. Then the graph is closed in $\mathbb{R}^n \times \mathbb{R}$ and Nash diffeomorphic to $M$. Since $\mathbb{R}^{n+1}$ is Nash diffeomorphic to $S^{n+1} - \text{a point}$ by the Stereographic projection, we can assume that the Zariski closure $\overline{M}$ in $\mathbb{R}^n$ is compact and that $\mathfrak{M}'$ is a point. Let
\( \lambda: M' \to \overline{M} \) be the normalization of \( \overline{M} \) (see [7]). Then there exists a Nash manifold \( M'' \) open in \( M' \) such that the restriction of \( \lambda \) to \( M'' \) is Nash diffeomorphic onto \( M \) and that \( M'' \) is a set of non-singular points of \( M' \). It follows that \( \exists M'' \subset \lambda^{-1}(\partial M) \) and that \( M' \) is compact because so is \( \overline{M} \). Apply Hironaka's desingularization theorem [2] to \( M' \). Then we have a compact non-singular affine algebraic set \( X \) of dimension \( n' \) and a smooth rational mapping \( \mu: X \to M' \) such that the restriction of \( \mu \) to \( \mu^{-1}(M'') \) is diffeomorphic onto \( M'' \). Moreover we can suppose that \( Z = \mu^{-1}(\lambda^{-1}(\partial M)) \) has only normal crossings (Main Theorem II in [2]). This means (ii). As \( \exists \mu^{-1}(M'') \subset Z \), \( \mu^{-1}(M'') \) is a connected component of \( X - Z \). Hence we can assume (i).

(2) Let \( p: V \to X \) be the orthogonal projection of a Nash tubular neighborhood \( V \) of \( X \) in \( \mathbb{R}^n \). Put

\[
Z' = Z \cap \overline{M},
\]

\[
F = \{(x, y) \in X \times \mathbb{R}^n | y \text{ is a normal vector of } X \text{ at } x \text{ in } \mathbb{R}^n \}.
\]

Then the projection \( F \to X \) shows that \( F \) is the normal bundle of \( X \) in \( \mathbb{R}^n \). It is easy to see that \( F \) is a non-singular algebraic set. Let \( F|_Y \) denote \( F \cap Y \times \mathbb{R}^n \), the restriction of the bundle to \( Y \), for any subset \( Y \) of \( X \).

We want to show the following. There exist a compact non-singular algebraic set \( Y \) in \( M \) of codimension 1, a connected component \( M' \) of \( M - Y \), a polynomial mapping \( q: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) and open neighborhoods \( U_1, U_2 \) of \( Y \times 0 \times 0 \) in \( F|_Y \times \mathbb{R} \) such that,
(i) \( q(x,0,0) = x \) for \( x \in Y \),

(ii) \( \bar{U}_1 \subset U_2 \),

(iii) \( q|_{\bar{U}_1} \) is a diffeomorphism into \( \mathbb{R}^n \) whose image contains \( \bar{M}' \).

(iv) \( q|_{U_2} \) is an immersion whose image contains \( \bar{M} - M' \).

Hence we can say that \( F|_Y \times \mathbb{R} \) and \( q(U_1) \) are the normal bundle of \( Y \) in \( \mathbb{R}^n \) and a "bent" tubular neighborhood respectively.

**Proof.** Let \( \mathfrak{a} \) be the ideal of the smooth rational function ring on \( X \) consisting of functions which vanish on \( Z \). Let \( \xi \) be the square sum of finite generators of \( \mathfrak{a}' \). Then for every point \( a \) of \( Z \), there exists an analytic local coordinate system \( (x_1, \ldots, x_n) \) for \( X \) centered at \( a \) such that \( \xi = x_1^2 \ldots x_n^2 \) in a neighborhood of \( a \) for some \( n' \). Put \( Y = \xi^{-1}(\epsilon) \cap M \) for sufficiently small \( \epsilon > 0 \). Here \( Y \) is not necessarily algebraic, so we approximate later it by an algebraic set.

For any point \( a \in Z' \), consider the set of all connected components of \( M \cap (a \text{ small ball with center at } a) \). Let \( T \) be the disjoint union of the set as \( a \) runs on \( Z' \). Hence an element \( c \) of \( T \) means a pair of a point \( \sigma_1(c) \) of \( Z' \) and a connected set \( \sigma_2(c) \) contained in \( M \). Then \( T \) has a topological manifold structure such that \( \sigma_1: T \to X \) is a topological immersion and that \( \sigma_2(c) \cap \sigma_2(c') \neq \emptyset \) for close \( c, c' \in T \). Let \( v_1: T \to \mathbb{R}^n \) be a continuous mapping which satisfies the following conditions. For every point \( c \) of \( T \), let \( (x_1, \ldots, x_n) \) be an analytic local coordinate system for \( X \) centered at \( a = \sigma_1(c) \) such that \( \sigma_2(c) = \{ x_1 > 0, \ldots, x_n > 0 \} \), \( n' \leq n' \), in a neighborhood of \( a \). Then \( v_1(c) \) is a vector tangent to \( X \) at \( a \) and satisfies
\[ v_1(c)x_i > 0 \text{ for } 1 \leq i \leq n, \]

here we regard \( v_1(c) \) as a tangent vector of \( X \) at \( a \). This means that \( v_1(c) \) points at a point of \( \sigma_2(c) \). The existence of \( v_1 \) is trivial. Moreover we can assume the following, using a \( C^\infty \) partition of unity. For every \( c \in T \), there exists a \( C^\infty \) vector field \( v_2(c) \) on a small neighborhood of \( a = \sigma_1(c) \) in \( \mathbb{R}^n \) such that \( v_2(c)_a = v_1(c) \) and that \( v_2(c') = v_2(c'') \) on the common domain of definition for any close \( c', c'' \in T \).

Put

\[ \sigma_2'(c) = p^{-1}\sigma_2(c) \text{ for } c \in T. \]

We remark that \( p^{-1}(Z) \) has only analytic normal crossings in \( V \) (see [2] for the definition) and that \( \sigma_2'(c) \) can be regarded as a connected component of \( p^{-1}(M) \cap (a \text{ small ball with center at } \sigma_1(c)) \), because we are concerned with only an arbitrarily small neighborhood of \( Z' \). Consider the restrictions of \( v_2(c) \) to \( \sigma_2'(c) \) for all \( c \in T \). Then the restrictions of \( v_2(c) \) and \( v_2(c') \) to \( \sigma_2'(c) \cap \sigma_2'(c') \) are equal for \( c, c' \in T \). Hence we have a \( C^\infty \) vector field \( v_3 \) on (a neighborhood of \( Z' \) in \( \mathbb{R}^n \)) \( \cap p^{-1}(M) \) such that \( v_3 = v_2(c) \) on \( \sigma_2'(c) \). By the property of \( v_1, v_3 \) is transversal to \( p^{-1}(Y) \) for any small \( \varepsilon > 0 \) (\( Y = \xi^{-1}(\varepsilon) \cap M \)).

Fix \( \varepsilon \). Using the integral curves of \( v_3 \), we obtain a \( C^\infty \) imbedding \( q_1 \) of a neighborhood \( U_1 \) of \( Y \times 0 \times 0 \) in \( F|_1 \times \mathbb{R} \) into \( \mathbb{R}^n \) such that

\[ q_1(x, y, 0) = x + y, \]

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\[ \frac{\partial q_1}{\partial t}(x,y,t) = v_3 q_1(x,y,t) \quad \text{for} \quad (x,y,0), (x,y,t) \in U, \]

and that \( q_1(U_1) \) is equal to (a neighborhood of \( Z' \) in \( \mathbb{R}^n \)) \( \cap p^{-1}(M) \). Here \( U_1 \) is chosen so that \((U_1; Y \times 0 \times 0)\) is \( C^\infty \) diffeomorphic to \((F|_Y \times \mathbb{R}, Y \times 0 \times 0)\). From these arguments it follows that \( M - Y \) has two connected components the closure of one of which does not intersect with \( \partial M \). Let the component be written as \( M' \). Then we can assume that \( q_1(U_1) \) contains \( M - M' \) and hence that \((M)\). Let \( q_2 \) be a \( C^\infty \) extension of \( q_1 \) to \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \). Then there exists an open neighborhood \( U_2 \) of \( Y \times 0 \times 0 \) in \( (F|M \times \mathbb{R} \) such that (ii) and (iv) are satisfied.

We need to approximate \( Y \) and \( q_2 \) by an algebraic set and a polynomial mapping. Since \( \bar{M}' \) is a \( C^\infty \) manifold with boundary, we have a \( C^\infty \) function \( \chi \) on \( X \) such that \( \chi \) is \( C^\infty \) regular on \( Y \) and that the zero set of \( \chi \) is \( Y \). Approximate \( \chi \) by a smooth rational function in the \( C^\infty \) topology, and consider the zero set. If we use the same notation \( Y \) for the set, \( Y \) is a compact non-singular algebraic set in \( M \) of codimension 1. We have no problem to apply the above argument to this \( Y \), because the old \( Y \) can be transformed to the new one by a \( C^\infty \) diffeomorphism of \( \mathbb{R}^n \) arbitrarily close to the identity.

By the equality

\[ q_2(x,0,0) = x \quad \text{for} \quad x \in Y, \]

we have polynomial functions \( v_1, \ldots, v_k \) on \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) and \( C^\infty \) mappings \( \rho_1, \ldots, \rho_k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \) such that

\[ q_2 = \sum_{i=1}^{k} v_i \rho_i + \text{the projection onto the first factor}, \]
and that \( v_1 = 0 \) on \( Y \times 0 \times 0 \). Approximate \( \rho_i \) by polynomial mappings \( \rho'_i \) in the compact-open \( C^\infty \) topology. Then

\[
q = \sum_{i=1}^{k} v_i \rho'_i + \text{the projection}
\]

is what we wanted. We have to modify \( U_1, U_2 \) so that (iii), (iv) remain valid. But this is easy to see, hence we omit it.

The diagram is inserted here.

(3) By (iii) in (2), \( q \) maps diffeomorphically \( (q^{-1}(M-M')) \cap U_1, Y \times 0 \times 0 \) onto \( (M-M', Y) \). The construction of \( Y \) and \( q \) in (2) shows that \( (q^{-1}(M-M')) \cap U_1, Y \times 0 \times 0 \) is \( C^\infty \) diffeomorphic to \( (Y \times (-1,0], Y \times 0) \). Hence \( M \) and \( M' \) are \( C^\infty \) diffeomorphic. We want to prove that they are Nash diffeomorphic. As it is not easy to prove directly this, we will use an intermediary Nash manifold \( N \) which shall be Nash diffeomorphic to \( M \) and \( M' \). In (3) we will define a \( C^\infty \) manifold \( M'' \) whose approximation shall be \( N \).

Let \( q': \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \) be the projection to the first factor. Put

\[
A = F \mid_{\mathbb{R}^n \times \mathbb{R}} , \quad \text{S= the critical points set of } \ q \mid_{F \mid_{\mathbb{R}^n \times \mathbb{R}^r}} \\
B = (A \cap q^{-1}(Z)) - S \quad \text{(where } \overset{\cdot}{=} \text{ means the Zariski closure),}
\]

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\[ C = (A \cap \mathcal{q}^{-1}(X)) \cdot S \text{ and } B' = B \cap \overline{U}_1. \]

Then \( A \) is a non-singular algebraic set, \( B \) and \( C \) are algebraic sets of dimension \( n'-1, n' \) respectively, and \( B' \) is a semi-algebraic set of dimension \( n'-1 \). Moreover \( B \) has only normal crossings in \( C \) at every point of \( B \cap U_2 \) (see (ii) in (1)), \( C \) is non-singular at every point of \( C \cap U_2 \), and for every point \( a \) of \( B' \) there exists an algebraic local coordinate system \((x_1, \ldots, x_n, t)\) for \( C \) centered at \( a \) such that

\[ B' = \{ x_1 = 0, x_2 \geq 0, \ldots, x_n \geq 0 \} \cup \ldots \cup \{ x_1 \geq 0, \ldots, x_n' \geq 0, x_n'' = 0 \} \]

in a neighborhood of \( a \) for some \( n'' \leq n' \) and that

(1) \( q' \) maps diffeomorphically \( \{ x_1 = 0 \}, \ldots, \{ x_n'' = 0 \} \) into \( Y \).

We remark that \( B' \) is naturally homeomorphic to \( T \) in (2). Put

\[ C' = q^{-1}(M-M') \cap U_1. \]

Then \( C' \) is the subdomain of \( C \) sandwiched in between \( B' \) and \( Y \times 0 \times 0 \).

We want to find a \( C^\infty \) manifold \( M'' \) in \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) and a \( C^\infty \) diffeomorphism \( \psi : M'' \to M \) such that

(ii) \( M'' \supset C' \), \( \psi = q \) on \( C' \), \( \overline{M''} \cap B = \overline{M''} = B' \) and \( M'' \cap C = C' \).

**Proof.** Since \( q \) maps \( (C' \cup Y \times 0 \times 0, Y \times 0 \times 0) \) diffeomorphically to \( (M-M', Y) \), we only have to find a compact \( C^\infty \) manifold \( M^{(3)} \) with boundary in \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) and a diffeomorphism \( \psi' : M^{(3)} \to \overline{M'} \).
such that

(iii) \( M(3) = Y \times 0 \times 0 \),

(iv) \( q = \varphi' \) on \( Y \times 0 \times 0 \),

(v) \( M(3) \cap C = Y \times 0 \times 0 \), and

(vi) \( M(3) \cup C' \) is a \( C^\infty \) manifold.

Let \( O_\varepsilon \) denote the \( \varepsilon \)-neighborhood of \( Y \times 0 \times 0 \) in \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) for small \( \varepsilon > 0 \). Let \( \chi_i \), \( i=1,2 \), be a \( C^\infty \) function on \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) such that

\[
0 \leq \chi_1 \leq 1, \quad \chi_1 = \begin{cases} 1 & \text{outside } O_{2\varepsilon} \\ 0 & \text{in } O_\varepsilon \end{cases}
\]

and that if \( \chi_1(x) \neq 1 \) then \( \chi_2(x) = 0 \). Consider the mapping

\[
\varphi'' : O_{3\varepsilon} \cap (C-C') \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}
\]

defined by

\[
\varphi''(z) = (1-\chi_2(z))(0,z_2,z_3) + \chi_1(z)(q(z)z_1+0,0)+(z_1,0,0), \quad z=(z_1,z_2,z_3).
\]

Take sufficiently small \( \varepsilon \). Then, choosing \( \chi_1 \) suitably we see that \( \varphi'' \) is a \( C^\infty \) diffeomorphism. It follows that

\[
\varphi''((O_{3\varepsilon} \cap O_{2\varepsilon}) \cap (C-C')) \subset M' \times 0 \times 0.
\]

Put

\[
M(3) = (M' - q(O_{3\varepsilon} \cap (C-C') \cap 0 \times 0 \times 0) \cup \varphi''(O_{3\varepsilon} \cap (C-C')).
\]

Then \( M(3) \) is a compact \( C^\infty \) manifold with boundary \( Y \times 0 \times 0 \) (iii).

Let \( \varphi'^{-1} : M' \to M(3) \) be defined by

\[
\varphi'^{-1}(x) = \begin{cases} \varphi''(q^{-1}(x) \cap O_{3\varepsilon} \cap (C-C')) & \text{if } x \in q(O_{3\varepsilon} \cap (C-C')) \\ (x,0,0) & \text{otherwise.} \end{cases}
\]
Then $\varphi^{-1}$ is a $C^\infty$ diffeomorphism such that $\varphi'=q$ in a neighborhood of $Y \times 0 \times 0$ (iv). From $\varphi''(0, \cap (C-C')) = 0, \cap (C-C')$, (vi) follows. For (v), we modify $M^{(3)}$ as follows. Increasing the dimension $n$ if necessary, we can assume that

$$X \subset \mathbb{R}^{n-1} \times 0,$$

and hence $C, M \subset \mathbb{R}^{n-1} \times 0 \times \mathbb{R} \times \mathbb{R}$.

Let $x_3$ be a $C^\infty$ function on $M^{(3)} \cup C'$ such that $x_3 = 0$ on $Y \times 0 \times 0 \cup C'$ and $>0$ on $M^{(3)} - Y \times 0 \times 0$. Consider

$$\{(x_1, x_3(x_1,0,y,t), y,t) | (x_1,0,y,t) \in M^{(3)}\}$$

in place of $M^{(3)}$. Then (v) is satisfied.

(4) Here we will approximate $M''$ by a Nash manifold $N$ fixing the "boundary". Let $L'$ be a small open semi-algebraic neighborhood of $B'$ in $C$, and $L$ be the union of $M''$ and $L'$ such that $\overline{L}$ is a $C^\infty$-manifold with boundary. This is possible since $C$ is non-singular at every point of $B'$. Let $D'$ be an open tubular neighborhood of $L$ in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, and $D$ be an open semi-algebraic subset of $D'$ containing $L$. We can choose $M'', L'$ and $D$ so that $D \cap C$ is a small neighborhood of $\overline{C'}$ in $C$ and that $D \cap B$ is equal to $L' \cap B$ and that $B$ has only normal crossings in $C$ at every point of $D \cap B$.

Let $r: D \rightarrow L$ denote the orthogonal projection. Let $h: D \rightarrow E_{m, n''}^{m, n''}$, $m=2n-n'+1$, be defined by

$$h(z) = (h_1(z), h_2(z)) =$$

(the normal vector space of $L$ at $r(z)$ in $\mathbb{R}^{2n+1}, z-r(z)$)
for $z \in D$. Then $h$ is a Nash map on $r^{-1}(L')$, and $h_2^{-1}(0) = L$.

**Remark 9.** Let $f : M_1 \to M_2$ be a $C^\infty$ mapping of Nash manifolds. Then we can approximate $f$ by Nash mappings in the compact-open $C^\infty$ topology (this is announced in [9]).

**Proof.** By Proposition 1 in [9] there exist a compact non-singular algebraic set $X_1 \subset \mathbb{R}^{n_1}$, a closed semi-algebraic subset $B_1'$ of $X_1$ and a union $M_1'$ of connected components of $X_1 - B_1'$ such that

(i) $M_1'$ is Nash diffeomorphic to $M_1'$,

(ii) for every point $x$ of $B_1'$, there exists an analytic local coordinate system $(x_1', ..., x_{n_1}')$ for $X_1$ centered at $x$ such that

$$(M_1', B_1') = \{(x_1 > 0, ..., x_{n_1}') > 0\},$$

$$(x_1 = 0, x_2 > 0, ..., x_{n_1}' > 0) \cup \cdots \cup \{(x_1 > 0, ..., x_{n_1}' - 1 > 0, x_{n_1}' = 0)\}$$

in a neighborhood of $x$, for some $n_1'' < n_1'$. Hence we can say that $M_1'$ is a compact analytic manifold with cornered boundary. We assume $M_1' = M_1$. It follows that $\partial M_1 = B_1'$.

In the same way as (2), we can construct a compact non-singular algebraic set $Y_1$ in $M_1 \cap$ (an arbitrarily small neighborhood of $\partial M_1$) and an analytic imbedding $q_1' : Y_1 \times [-1, 0] \to X_1$ such that $q_1'(Y_1 \times 0) = Y_1$ and that the image of $q_1'$ is an arbitrarily small neighborhood of $B_1'$. Put

$$M_1'' = q_1'(Y_1 \times [-1, 0]) \cup M_1.$$ 

Then $M_1''$ is a compact analytic manifold with boundary containing
and there exists a $C^\infty$ diffeomorphism $\pi$ of $X_1$ arbitrarily close to the identity such that $\pi(M') \subset M_1$.

Let $M_2$ be contained in $\mathbb{R}^{n_2}$, and $p$ be the orthogonal projection of a Nash tubular neighborhood of $M_2$ in $\mathbb{R}^{n_2}$ (Lemma 7). Consider $f \circ \pi$ on $M'$. Then $f \circ \pi$ is extensible to $X_1$ and hence to $\mathbb{R}^{n_1}$ as a $C^\infty$ mapping to $\mathbb{R}^{n_2}$. Let $\eta$ be an extension, and $\eta'$ be a polynomial approximation of $\eta$. Then $f' = \eta'|_{M_1}: M_1 \to \mathbb{R}^{n_2}$ is an approximation of $f: M_1 \to \mathbb{R}^{n_2}$. Since the closure of $\pi(M_1)$ in $X_1$ is compact, we can assume that $f'(M_1)$ is contained in the Nash tubular neighborhood of $M_2$. Hence $p \circ f': M_1 \to M_2$ is a Nash approximation of $f$. Thus Remark is proved.

In many cases we want Nash approximation to be fixed on a given semi-algebraic set. So the following are useful.

**Lemma 10.** For any $C^\infty$ function $g$ on $D$ vanishing on $D \cap B$, there exist $C^\infty$ functions $\alpha_1, \ldots, \alpha_\ell$ and Nash functions $\beta_1, \ldots, \beta_\ell$ on $D$ such that

$$g = \alpha_1 \beta_1 + \ldots + \alpha_\ell \beta_\ell$$

$$\beta_1 = \ldots = \beta_\ell = 0 \quad \text{on} \quad D \cap B.$$

**Proof.** Let $p$ be the ideal of the smooth rational function ring on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ consisting of functions which vanish on $B$. Let $\beta_1, \ldots, \beta_\ell$ be a system of generators of $p$. We want to find $\alpha_1, \ldots, \alpha_\ell$ so that the equality in Lemma is satisfied for these $\beta_1, \alpha_1$. By a $C^\infty$ partition of unity we only need to
see this locally. This is trivial in a neighborhood of any point of $D-B$.

For any point $a$ of $D \cap B$, there exist smooth rational functions $\gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_n$ with $k=2n+1-n'$ and for some $n'' \leq n'$ such that $\gamma_1, \ldots, \gamma_k$ vanish on $C$, that $\delta_1 \ldots \delta_n''$ vanishes on $B$, that

$$B = \{ \gamma_1 = \ldots = \gamma_k = \delta_1 \ldots \delta_n'' = 0 \}$$

in a neighborhood of $a$ and that

$$\gamma_1 \times \ldots \times \gamma_k \times \delta_1 \times \ldots \times \delta_n'': \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{k+n''}$$

is a submersion in a neighborhood of $a$, since $C$ is non-singular at $a$, and $B$ has only normal crossings at $a$ in $C$.

Hence it is sufficient to prove that if

$$D = \mathbb{R}^{k''} = \{ (x_1, \ldots, x_{k''}) \} \text{ and } B = \{ x_1 = \ldots = x_k = x_{k+1} \ldots x_{k'} = 0 \}$$

with $k \leq k' \leq k''$, and if a $C^\infty$ function $g$ on $\mathbb{R}^{k''}$ vanishes on $B$, then

$$g = \alpha_1 x_1 + \ldots + \alpha_k x_k + \alpha_{k+1} x_{k+1} \ldots x_{k'}$$

for some $C^\infty$ functions $\alpha_1, \ldots, \alpha_{k+1}$. The case when $k=0$ is trivial. Hence, considering $g(0, \ldots, 0, x_{k+1}, \ldots, x_{k''})$, we have a $C^\infty$ function $\alpha_{k+1}$ on $\mathbb{R}^{k''}$ such that $g = \alpha_{k+1} x_{k+1} \ldots x_{k'}$ on $0 \times \mathbb{R}^{k''-k}$. This implies that $g - \alpha_{k+1} x_{k+1} \ldots x_{k'}$ vanishes on $0 \times \mathbb{R}^{k''-k}$. Then the existence of $\alpha_1, \ldots, \alpha_k$ which satisfy
\[ g = \alpha_{k+1} x_{k+1} \cdots x_k = \alpha_1 x_1 \cdots + \alpha_k x_k \]

is well-known. Hence Lemma follows.

Lemma 11. With the same \( g \) as in Lemma 10, there exists a Nash function \( g' \) on \( D \) arbitrarily close to \( g \) and vanishing on \( D \cap B \).

Proof. Using Remark 9, we approximate \( \alpha_1 \) in Lemma 10 by Nash functions \( \alpha'_1 \). Then \( g' = \sum_{i=1}^{l} \alpha'_i \beta_i \) is a Nash approximation of \( g \) and vanishes on \( B \cap D \).

We continue with the construction of \( N \). By Remark 9 we have a Nash mapping \( h'_1 : D \to G_{m,n'} \) which is an approximation of \( h_1 \). Apply Lemma 11 to \( h_2 \). Then we have a Nash approximation \( h'_2 : D \to \mathbb{R}^{2n+1} \) of \( h_2 \) such that \( h'_2 = 0 \) on \( D \cap B \). Let \( W \) be a Nash tubular neighborhood of \( E_{m,n'} \) in \( \mathbb{R}^{n''} \times \mathbb{R}^{2n+1} \) where \( G_{m,n'} \) is naturally imbedded in \( \mathbb{R}^{n''} \) for some \( n'' \). Let \( s : W \to E_{m,n'} \) be the orthogonal projection. Put

\[ h'' = (h''_1, h''_2) = s \circ h' = s \circ (h'_1, h'_2). \]

Then \( h'' : D \to E_{m,n'} \) is a Nash approximation of \( h \), and \( h''_2 \) is identical to \( h_2 \) on \( D \cap B \). Shrinking \( L \) and \( D \) if necessary, we take this approximation in the uniform \( C^\infty \) topology. Put

\[ L'' = h''^{-1}(G_{m,n'} \times 0) = h''^{-1}(0). \]

Then there exists a \( C^\infty \) diffeomorphism \( \psi \) from \( L'' \) to \( L \).
close to the identity such that $\psi=\text{identity on } D \cap B$, because $h$ is transversal to $G_{m,n',x_0}$ in $E_{m,n}$. Put $\psi^{-1}(\mathbb{M})=\mathbb{N}$. It follows that $L''$ is a Nash manifold containing $D \cap B$ and that $(\mathbb{M}, B')$ is $C^\infty$ diffeomorphic to $(\mathbb{N}, B')$ identically on $B'$. Hence $N$ is the required Nash manifold.

(5) We will prove that $M$ and $N$ are Nash diffeomorphic. Let $\phi:L \to X$ be the $C^\infty$ extension of the diffeomorphism $\varphi:M'' \to M$ to $L$ such that $\phi(z)=q(z)$ for $z \notin M''$. Let $\psi:D \to \mathbb{R}^n$ be a $C^\infty$ extension of $\phi \circ \psi:L'' \to X$ to $D$. Then $\psi=q$ on $D \cap B$, and $\psi|_{L''}$ is an immersion. Apply Lemma 11 to $\psi-q$. Then we obtain a Nash approximation $\psi'$ of $\psi$ such that $\psi'=q$ on $D \cap B$. Compose $\psi'|_{L''}$ with the orthogonal projection $p$ of a Nash tubular neighborhood of $X$ in $\mathbb{R}^n$. This is well-defined if we shrink $L$ and $D$ and if the approximation is chosen closely. Then the composed function $\psi'':L'' \to X$ is an approximation of $\psi|_{L''}=\phi \circ \psi:L'' \to X$ such that $\psi''=q$ on $D \cap B=L'' \cap B$. Moreover we see $\psi''(N)=M$ as follows from the facts $\psi''(B')=q(B')=Z'$, that $M$ is a connected component of $X-Z'$ and that $\psi''|_{\mathbb{N}}$ is an immersion. It is trivial that $M \cap \psi''(N)$ is an open subset of $M$. Assume it to be not closed. Then there exists a convergent sequence of points $x_1, x_2, \ldots$ in $\psi''(N)$ whose limit $x \in M$ is not contained in $\psi''(N)$. Let $z_1, z_2, \ldots$ be points of $N$ such that $\psi''(z_i)=x_i$, $i=1, \ldots$. Choosing a subsequence, we can assume that $z_1, z_2, \ldots$ converges to $z \in \mathbb{N}$. Then we have $\psi''(z)=x$. This is a contradiction. Hence $\psi''(N) \supset M$. In the same way as above, we see that the set

$$\{ x \in M \mid \# \psi''|_{\mathbb{N}}^{-1}(x) \geq 2 \}$$
is empty or equal to \( M \). For any point \( x \in M \), if we choose the
above approximation closely, this set does not contain \( x \).
Hence \( \Phi''|_N \cap \Phi''^{-1}(M) \) is diffeomorphic onto \( M \). From the same
reason it follows that any connected component of \( X-Z'-M \) does
not contain any point of \( \Phi''(N) \), namely that \( \Phi''(N) \subset M \cup Z' \).
Then \( \Phi''(N) \cap Z' = \emptyset \). Hence \( \Phi''|_N \) is a Nash diffeomorphism onto
\( M \).

(6) Finally we will prove that \( M' \) and \( N \) are Nash diffeomorphi
For any point \( x \in L'' \cap B = D \cap B \), let \( C_x^\omega(L'') \) denote the ring of \( C^\omega \)
function germs at \( x \) in \( L'' \). Then the ideal \( q_x \subset C_x^\omega(L'') \) of
germs vanishing on \( L'' \cap B \) is principal because of the normal
crossings property of \( B \) in \( C \cap D \). Moreover we have a polynomial
function on \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) which vanishes on \( D \cap B \) and the germ of
whose restriction to \( L'' \) is a generator of \( q_x \). Choose \( D \) so
small that the fundamental class of \( L'' \cap B \) is mapped to the zero
class in \( H_{n-1}(L''; \mathbb{Z}_2) \) by the inclusion map (this is possible
since \( N \) is a compact topological manifold with boundary \( B' \),
here we use infinite chain). Then we see easily that the ideal \( q \)
of \( C_x^\omega(L'') \), the ring of \( C^\omega \) functions on \( L'' \), of functions
vanishing on \( L'' \cap B \) is principal (see Lemma 1 in [12]). Approx-
imate a generator of \( q \) by a Nash function \( \rho \) by the method
of Lemma 11 so that \( \rho = 0 \) on \( L'' \cap B \). Then it follows that \( \rho \)
generates \( q \). Choose \( \rho \) so that \( \rho > 0 \) on \( N \).

Recall \( q' : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \), the projection to the first
factor. The restriction of \( q' \) to \( B' \) is homeomorphic onto
\( Y \). Let us extend this to \( \bar{N} \) so that the extension maps diffeo-
morphically \( \bar{N} \) to \( M' \). Let \( v \) be the unit normal vector field
of \( Y \) in \( X \) pointing to the interior of \( M' \). Choose small \( L' \).
Put

$$\theta(z) = p \circ (q' \circ \psi(z) + \rho(z) v \circ q' \circ \psi(z)) \quad \text{for} \quad z \in \psi^{-1}(L').$$

Here we regard $v$ as a mapping from $Y$ to $\mathbb{R}^n$, and $p$ is the orthogonal projection of a Nash tubular neighborhood of $X$ in $\mathbb{R}^n$. This is well-defined because $q'(z) \in Y$ for $z \in L'$. Clearly $\theta$ is a Nash mapping.

We can assume that $L' \cap M'' = \{ z \in M'' \mid p \circ \psi^{-1}(z) < \varepsilon \}$ for some $\varepsilon > 0$. Let $v'$ be the unit $C^\infty$ vector field on $L'$ the family of whose maximal integral curves consists of $\{ q'^{-1}(x) \cap L' \}_{x \in Y}$ and which points into $M''$ at every point of $B'$. For any point $a \in B'$, there exists a local analytic coordinate system $z = (z_1, \ldots, z_n)$ for $L'$ centered at $a$ such that $p \circ \psi^{-1}(z) = z_1 \ldots z_n$ for some $n'' < n$ and that $M'' = \{ z_1 > 0, \ldots, z_n'' > 0 \}$ in a neighborhood of $a$. By (i) in (3), $v' z_i > 0$, $i = 1, \ldots, n''$ at $a$. It follows that $v' \circ p \circ \psi^{-1} > 0$ on $L' \cap M''$. Hence $v'$ is transversal to $\{ z \in M'' \mid p \circ \psi^{-1}(z) = \varepsilon' \}$ for some $\varepsilon' > 0$. This implies that $q'$ maps $\{ z \in M'' \mid p \circ \psi^{-1}(z) = \varepsilon' \}$ diffeomorphically onto $Y$. Therefore $\theta |_{\psi^{-1}(L')}$ is diffeomorphic onto $\psi^{-1}(L')$. The transversality of $v'$ shows also that $(M'' - L', \partial(M'' - L')$ is diffeomorphic to $(M'' - C', \partial(M'' - C'))$ so that if the diffeomorphism maps a point $z \in \partial(M'' - L')$ to $z' \in \partial(M'' - C')$ we have $q(z) = q(z')$. Hence there exists a diffeomorphism from $(M'' - L', \partial(M'' - L')$ to $(\bar{M}', Y)$ whose restriction to $\partial(M'' - L')$ is $q'$. Therefore we extend $\theta$ to $\theta : L'' \to X$ such that $\theta |_N$ is a $C^\infty$ diffeomorphism onto $M'$.

Apply Lemma 11 to $\theta - q'$, and compose (the approximation mapping+$q'$) with $p$. Then we have a Nash approximation $\theta' : L'' \to X$ of $\theta$ such that $\theta' = \theta$ on $L'' \cap B$. To see that $\theta' |_N$ is a Nash diffeomorphism onto $M'$ we only need to show the following by the same reason as (5).
(i) $\theta'(B')=q'(B')=Y$.

(ii) $M'$ is a connected component of $X-Y$.

(iii) $\theta'\big|_N$ is an immersion.

(i) and (ii) have been shown already. It is trivial that $\theta'\big|_{N-\psi^{-1}(L')}\big|_N$ is an immersion. Hence we only have to prove the following.

Statement: Let $\theta_1: \mathbb{R}^n' \rightarrow \mathbb{R}^{n'-1}$ be a submersion, $K \subset \mathbb{R}^{n'}$ be a compact set. Let $v_1$ be a unit $C^\infty$ vector field on $\mathbb{R}^{n'}$ the family of whose all maximal integral curves consists of $\{v_1(y)\}_{y \in \mathbb{R}^{n'-1}}$. Put $\theta_2(x)=x_1\ldots x_{n''}$ for $x=(x_1,\ldots,x_{n'}) \in \mathbb{R}^{n'}$.

Assume that $v_1 x_i>0$, $i=1,\ldots,n''(\leq n')$. Let $(\theta_1',\theta_2')$ be a $C^\infty$ close approximation of $(\theta_1,\theta_2)$ such that $\theta_2'^{-1}(0) \subset \theta_2^{-1}(0)$. Then $(\theta_1',\theta_2')$ is an immersion on $\{x_1>0,\ldots,x_{n''}>0\} \cap K$.

Proof of Statement. Since $\theta_2'$ vanishes on $\{x_1=0\} \cup \ldots \cup \{x_{n''}=0\}$, there exists a $C^\infty$ function $\eta$ on $\mathbb{R}^{n'}$ such that $\theta_2'=\eta\theta_2$. We see easily that $\eta$ is close to the function 1 (see the statement at p. 268 in [10]). Replacing $\eta$ by a $C^\infty$ function which is equal to $\eta$ in a neighborhood of $K$ and is close to 1 in the Whitney $C^\infty$ topology, we can assume that

$$\frac{\partial (\eta x_1)}{\partial x_1} > 0 \quad \text{on } \mathbb{R}^{n'}.$$ 

Then $\pi:(x_1,\ldots,x_n) \rightarrow (\eta x_1,x_2,\ldots,x_n)$ is a $C^\infty$ diffeomorphism close to the identity. Since $\theta_2'\circ \pi^{-1}=\theta_2$ and $\pi\{x_1>0,\ldots,x_{n''}>0\} =\{x_1>0,\ldots,x_{n''}>0\}$, we need to treat only the case where $\theta_2' = \theta_2$. By the same reason as above, we can assume that $\theta_1'$ is sufficiently close to $\theta_1$ in the Whitney $C^\infty$ topology. Then $\theta_1'$ is
a submersion. Let $v_1$ be the unit vector field on $\mathbb{R}^n'$ the family of whose all maximal integral curves consists of \( \{ \theta_1^{-1}(y) \}_{y \in \mathbb{R}^{n'-1}} \) and which is close to $v_1$. Then we have $v_1'x_i > 0$, $i=1, ..., n''$. Hence

\[ v_1'(x_1 \ldots x_{n''}) > 0 \quad \text{on} \quad \{ x_1 > 0, \ldots, x_{n''} > 0 \}. \]

This means that the Jacobian matrix of $(\theta_1', \theta_2')$ has the rank $n'$ on $\{ x_1 > 0, \ldots, x_{n''} > 0 \}$. We complete the proofs of Statement and hence of Theorem 1.

Proof of Theorem 2. Let $N_1$, $N_2$ be contained in non-singular algebraic sets $X_1$, $X_2 \subset \mathbb{R}^n$ respectively so that $\exists N_1 = Y_1$ and $\exists N_2 = Y_2$ are non-singular. The implication $(ii) \Rightarrow (i)$ is trivial.

First we will prove $(i) \Rightarrow (iii)$. Let $\varphi_1$, $\varphi_2$ be Nash functions on $X_1$, $X_2$ respectively such that $\varphi_1^{-1}(0) = Y_1$, $\{ \varphi_1 > 0 \} = M_1$ and that $\varphi_1$ are $C^\infty$ regular at $Y_1$. The existence of such $\varphi_1$ follows from the non-singular property of $Y_1$ (see Lemma 1 in [12]) (in fact, we can choose as $\varphi_1$ polynomial functions). Let $\phi_1$, $\phi_2$ be positive proper Nash functions on $M_1$ defined by

\[ \phi_1 = 1/(\varphi_1|_{M_1}), \quad \phi_2 = (1/\varphi_2|_{M_2})^{\circ \tau}, \]

where $\tau: M_1 \to M_2$ be a Nash diffeomorphism. Apply Lemma 8 to $\phi_1$ and $\phi_2$. Then there exists a $C^\infty$ diffeomorphism $\pi$ of $M_1$ such that $\phi_1$ and $\phi_2^{\circ \tau \circ \pi}$ are equal outside a compact subset of $M_1$. Hence we have $\varphi_1 = \varphi_2^{\circ \tau \circ \pi}$ on (a neighborhood of $\exists M_1$ in $M_1$). This means that $\tau \circ \pi$ maps $\{ \varphi_1 = \varepsilon \}$ to $\{ \varphi_2 = \varepsilon \}$ for small $\varepsilon > 0$. Hence the restriction of $\tau \circ \pi$ on $\{ \varphi_1 \geq \varepsilon \}$ for
small $\varepsilon > 0$ is a $C^\infty$ diffeomorphism onto $\{\varphi_2 \geq \varepsilon\}$. As $\{\varphi_1 \geq \varepsilon\}$, $\{\varphi_2 \geq \varepsilon\}$ are $C^\infty$ diffeomorphic to $N_1$, $N_2$ respectively, $N_1$ and $N_2$ are $C^\infty$ diffeomorphic.

We prove the inclusion (iii) $\Rightarrow$ (ii) in the next general form.

Lemma 12. Let $L_1 \supset L_2$, $L_1' \supset L_2'$ be compact Nash manifolds with or without boundary and compact Nash submanifolds. Assume $L_2 = \emptyset L_1$ if $L_2 \cap \emptyset L_1 \neq \emptyset$. If there is a $C^\infty$ diffeomorphism from $(L_1, L_2)$ to $(L_1', L_2')$, we can approximate it by Nash one. If the restriction of the given diffeomorphism to $L_2$ is of Nash class, the approximation can be chosen to take the same image as the diffeomorphism at each point of $L_2$.

Proof. The idea of the proof is the same as in Proof of Theorem 1, and this proof is easier than that since $L_2$ and $L_2'$ are smooth. Hence we give only the sketch. The case where $L_1$, $L_1'$ have the boundaries: Consider their doubles $L_3$, $L_3'$, and give them Nash structures [7]. We approximate the natural respective imbeddings of $L_1$, $L_1'$ into $L_3$, $L_3'$ by Nash mappings. Then we can regard $L_1$, $L_1'$ as contained in $L_3$, $L_3'$ respectively, and there is a $C^\infty$ diffeomorphism from $(L_3, L_1, L_2)$ to $(L_3', \emptyset L_1 \cup L_2^1)$. If we can approximate the induced diffeomorphism from $(L_3, \emptyset L_1 \cup L_2)$ to $(L_3', \emptyset L_1' \cup L_2')$ by a Nash one, Lemma 12 follows. Hence we can assume that $L_1$, $L_1'$ have no boundary. Here we do not necessarily assume that $L_2$ has the global dimension, namely that the local dimension is constant.

Assume that $L_2$ is connected for the sake of brevity. Let $\pi : (L_1, L_2) \rightarrow (L_1', L_2')$ be a $C^\infty$ diffeomorphism. If $\pi|_{L_2}$ is not of Nash class, by Remark 9 we approximate $\pi|_{L_2}$ by a Nash diffeomorphism $\pi' : L_2 \rightarrow L_2'$. Choose $\pi'$ very closely.
Then we easily find a \( C^\infty \) extension \( \pi'': L_1 \to L'_1 \) of \( \pi' \) such that \( \pi'' \) is an approximation of \( \pi \). Hence, from the beginning we can assume that \( \pi|_{L_2} \) is of Nash class. Let \( L_1, L'_1 \) be contained in \( \mathbb{R}^n, \mathbb{R}^{n'} \) respectively, and \( p : \mathbb{R}^n \times \mathbb{R}^{n'} \to \mathbb{R}^n \), \( p' : \mathbb{R}^n \times \mathbb{R}^{n'} \to \mathbb{R}^{n'} \) be the projections. Let \( L''_2 \subset \mathbb{R}^{n+n'} \) denote the graph of \( \pi|_{L_2} \). Then \( L''_2 \) is a Nash manifold such that \( p|_{L''_2}, p'|_{L''_2} \) are Nash diffeomorphisms onto \( L_2, L'_2 \) respectively. By the normalization of the Zariski closure \( \overline{L''_2} \) of \( L''_2 \), there exist a non-singular connected component \( S_2 \) of an algebraic set in \( \mathbb{R}^{n''} \) and a linear mapping \( \varphi \) from \( \mathbb{R}^{n''} \) to \( \mathbb{R}^{n+n'} \) such that \( \varphi|_{S_2} \) is diffeomorphic onto \( L''_2 \). Increasing \( n'' \) if necessary, we construct a \( C^\infty \) manifold \( S_1 \) in \( \mathbb{R}^{n''} \) and \( C^\infty \) diffeomorphisms \( \psi : S_1 \to L_1, \psi' : S_1 \to L'_1 \) such that \( S_2 \subset S_1 \), \( \psi=p\varphi \) on \( S_2 \), \( \psi'=p'\varphi \) on \( S_2 \) and \( \overline{S_2} \cap S_1 = S_2 \). Using \( E_{m,m'} \) in the same way as Proof (4) of Theorem 1, we reduce \( S_1 \) to a Nash manifold. Then we can find in the same way as Proofs (5), (6) Nash approximations \( \psi : S_1 \to L_1, \psi' : S_1 \to L'_1 \) of \( \psi', \psi' \) such that \( \psi=\psi, \psi'=\psi' \) on \( S_2 \). Hence \( \psi'\circ \psi^{-1} : L_1 \to L'_1 \) is a Nash approximation of \( \pi \) such that \( \psi'\circ \psi^{-1} = \pi \) on \( L_2 \). Lemma is proved.

4. Proofs of Corollaries

Proof of Corollary 3. Let \( N_1, N_2 \) be the compactifications of \( M_1, M_2 \) respectively. By Theorem 2, we only have to prove that \( N_1 \) and \( N_2 \) are \( C^\infty \) diffeomorphic. Let \( L \) be
a closed $C^\infty$ collar of $N_1$. Put $L' = N_1-L$, $L'' = \bar{L}'-L'$.
Let $\pi: M_1 \rightarrow M_2$ be a $C^\infty$ diffeomorphism. We see easily that
$(N_2-\pi(L')); \partial N_2, \pi(L''))$ is a $C^\infty$ h-cobordism. On the other
hand it follows from the assumption that $\partial N_2$ is simply
connected for $\dim M_1 \geq 6$. Hence, by the h-cobordism theorem
$N_2-\pi(L')$ is diffeomorphic to $\partial N_2 \times [0,1]$. This means that
there exists a homeomorphism $\tau: N_1 \rightarrow N_2$ such that $\tau|_L$
and $\tau|_{L'\cup L''}$ are $C^\infty$ diffeomorphic. It is easy to modify
$\tau$ to be a $C^\infty$ diffeomorphism. Hence Corollary 3 is proved.

**Proof of Corollary 4.** The correspondence is trivially injective
by Theorems 1,2.

**Surjectivity:** Let $N$ be a compact $C^\infty$ manifold with or
without boundary. We need to give to $N$ a Nash manifold structure.
If $N$ has the boundary, consider the double $N'$, and regard $N$
as naturally contained in $N'$. In the other case, put $N'=N$,
$\partial N=\emptyset$. Then, by a theorem in [1], $(N',\partial N)$ is $C^\infty$
diffeomorphic to a pair (an affine non-singular algebraic set, a non-singular
algebraic subset). By this diffeomorphism, we give to $N'$ an
algebraic structure. Then, since $N-\partial N$ is a union of connected
components of $N'-\partial N$, $N-\partial N$ is a Nash manifold. Obviously $N$
is the compactification of $N-\partial N$. Hence Corollary follows.

**Proof of Corollary 5.** Let $N_1$ be the compactification of $M_1$.
Obviously we can assume that $f$ is extensible to $N_1$ and hence
to the double of $N_1$. Consider a Nash manifold structure on the
double and a Nash imbedding of $M_1$ into it. Then, from the
beginning we can assume that $M_1$ is compact. Let $M_2$ be containd
in $\mathbb{R}^n$, and $q$ be the orthogonal projection of a Nash tubular neighborhood of $M_2$ in $\mathbb{R}^n$. Regard $f$ as a mapping to $\mathbb{R}^n$. If we can approximate $f$ by a Nash mapping $f':M_1 \to \mathbb{R}^n$ so that $f=f'$ on $M_1$, then $q \circ f':M_1 \to M_2$ is a required Nash approximation of $f$. Hence it is sufficient to consider the case of $M_2=\mathbb{R}$. We regard $\mathbb{R}$ as $S^1\{-\text{a point}\}$. Let $L \subset M_1 \times S^1$ be the graph of $f$. Put $L'=M_1 \times \{b\}$ where $b$ is a point of $S^1$. Then there exists a $C^\infty$ diffeomorphism $\pi$ of $M_1 \times S^1$ such that

$$\pi(x,b) = (x,f(x)) \quad \text{for } x \in M_1.$$ 

It follows that $\pi(L')=L$ and that $\pi|_{M_1 \times \{b\}}$ is of Nash class. Apply Lemma 12 to

$$\pi:(M_1 \times S^1,M_1 \times \{b\}) \to (M_1 \times S^1,\pi(M_1 \times \{b\})).$$

Then we obtain a Nash approximation $\tau$ of $\pi$ such that $\tau = \pi$ on $M_1 \times \{b\}$. For every point $x \in M_1$, put

$$g(x) = p \circ \tau(x,b)$$

where $p:M_1 \times S^1 \to S^1$ be the projection onto the second factor. Then $g$ is what we want.

Proof of Corollary 6. This corollary follows from Lemma 12 and the fact that $\mathbb{R}^n$ is Nash diffeomorphic to $S^n\{-\text{a point}\}$ and
that \((S^n, M)\) is \(C^\infty\) diffeomorphic to (an affine algebraic set, a non-singular algebraic subset)[1].

5. An example

Let \(W, W'\) be compact \(C^\infty\) manifolds with boundary such that the interiors are \(C^\infty\) diffeomorphic, but \(W\) and \(W'\) are not diffeomorphic (see Theorem 3 in [4]). Let \(X, X'\) be the doubles of \(W, W'\) respectively. We regard \(W, W'\) as naturally contained in \(X, X'\) respectively. By a theorem in [1] we can assume that \(X, X' \supseteq W\) and \(X' \supseteq W'\) are all non-singular algebraic sets in \(\mathbb{R}^n\). Let \(P, P'\) be polynomials on \(\mathbb{R}^n\) such that

\[
P^{-1}(0) = \partial W, \quad P'^{-1}(0) = \partial W'.
\]

Put

\[
Y = \{(x, y) \in X \times \mathbb{R} | yP(x) = 1\},
\]

\[
Y' = \{(x, y) \in X' \times \mathbb{R} | yP'(x) = 1\}.
\]

Then \(Y\) and \(Y'\) are \(C^\infty\) diffeomorphic non-singular affine algebraic sets, and their compactifications are the disjoint unions of 2 copies \(W+W (\subseteq X+X)\) and \(W'+W' (\subseteq X'+X')\) respectively. Hence, by Theorem 2, \(Y\) and \(Y'\) are not Nash diffeomorphic. Here it is not essential that \(Y, Y'\) are not connected. In fact we can find connected examples.

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References


