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<th>Title</th>
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SPECTRAL THEORY ON UNIFORMLY DISTRIBUTED SEQUENCES

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1. INTRODUCTION.

We shall consider sequences of real numbers or of complex numbers. The difference between simply normality and normality is to consider the frequency of any kind of block $B_k$. For the sequence of real numbers modulo 1 the corresponding relation is like between uniform distributed sequences and completely uniformly distributed sequences.

Theory of the uniform distribution of sequences is a fruitful branch in mathematics where number theory, probability theory and harmonic analysis meet with. Taking a normal number $x$ to base $q$, then the sequence $(q^n x)$ is uniformly distributed modulo 1. Hence we start from a modification of uniformly distributed sequences modulo 1. Then we discuss the spectrum and the spectral measure of a given sequence. These notions are from Fourier analysis and shall explain certain properties of the sequence considered.
2. NORMAL SETS.

Let \( \Lambda = ( \lambda_n ) \) be an infinite sequence of real numbers. A real number \( x \) is called \( \Lambda \)-normal if the sequence \( x\Lambda = ( x_\lambda_n ) \) is uniformly distributed modulo 1. Taking \( \Lambda = ( q^n ) \), then a \( \Lambda \)-normal number coincides with a usual normal number to base \( q \). The set of \( \Lambda \)-normal numbers is denoted by \( B(\Lambda) \). A set \( E \) of real numbers is called normal if there exists a sequence \( \Lambda \) such that \( E = B(\Lambda) \).

Mendès France and his colleagues collaborated on this notion (7), (8), (9) and (10) just after his arrival to Bordeaux University. Examples of normal sets are the followings:

i) the complement of any real algebraic field of finite degree;

ii) \( m\mathbb{Z}^* \) with positive integer \( m \);

iii) \( \mathbb{Q}^* \).

This series of papers ended with Gérard Rauzy's paper which gave a complete characterization of the normal set:

THEOREM ( RAUZY ). A set \( B \) is normal if and only if

i) \( 0 \notin B \);

ii) for every positive integer \( q \), \( qB \subseteq B \);

iii) there exist continuous real functions \( f_1, f_2, \ldots \) such
that $\lim_{n \to \infty} f_n(x) = 0$ exactly for those $x$'s which are elements of $B$.

The remaining unsolved (?) problems on normal sets are as follows:

1) Characterize the number $\mu$ such that $B + \mu = B$ implies the normality of $B$. (Every positive integer satisfies this property.)

2) A finite union of normal sets is normal or not.

In early 70's Mendès France considered the "depth" of a rational number using continued fraction expansions (13) and (14) and mentioned a relation to the formal language theory (16) and corresponding results on real algebraic numbers of second degree (remind Lagrange's theorem on periodic continued fractions) were proved by Henri Cohen and also by Peysant Leroux. Another related his paper to continued fractions is (24).

3. SPECTRUMS AND SPECTRAL MEASURES.

In analyzing a sequence of statistical data, the autocorrelation functions are the main powerful tool. This idea is back to Nöbert Wiener in 1927 for time series analysis. The progress of Fourier analysis allows us to apply this idea on
arithmetical sequences.

From now on we considered arithmetical sequences \( a = (a(n)) \) of complex numbers. The correlation function of

with lag \( h \) is defined by

\[
\gamma_a(h) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a(n) \overline{a(n+h)},
\]

where \( \overline{z} \) is the complex conjugate of \( z \). M.Bass defined a pseudo-random sequence \( a = (a(n)) \) satisfying following three conditions:

i) For each \( h \), \( \gamma_a(h) \) is determined;

ii) \( \gamma_a(0) \neq 0 \);

iii) \( \lim \limits_{h \to \infty} \gamma_a(h) = 0 \).

Mendès France proved the next theorem in his thesis (5):

THEOREM (MENDES FRANCE). If a real number \( x \) is normal to the base \( r \), then the sequence

\[
n \mapsto r_{\lfloor n \rfloor}(x) = \exp(2\pi i \lfloor x \cdot r \lfloor n \rfloor \rfloor / r)
\]

is pseudo-random and its correlation function is

\[
\gamma(h) = \begin{cases} 
1 & h = 0 \\
0 & \text{otherwise} 
\end{cases}
\]

A normal number is a model of a random sequence, hence this theorem seems quite natural.

By replacing iii) by
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{h=1}^{N} |\gamma(h)|^2 = 0, \]

we call \( \alpha = (\alpha(n)) \) is pseudo-random in the sense of Bertrandias. From now on we used "pseudo-random" in the sense of Bertrandias.

Let us introduce another notion: spectrum of a sequence \( \alpha = (\alpha(n)) \):

\[ \text{sp}(\alpha) = \{ \lambda \in [0,1); \lim_{N \to \infty} \sup_{N} \frac{1}{N} \sum_{k=1}^{N} \alpha(k) \exp(-2\pi i \lambda k) \leq \} \]

Van der Corput difference theorem said that pseudo-random sequence has empty spectrum. The converse is not in general true. But in certain class of sequences, these two notions are equivalent. Indeed, for any q-multiplicative sequence empty spectrum implies pseudo-random (19). An arithmetical function \( \alpha \) is called q-multiplicative if

\[ \alpha(\alpha q^k + b) = \alpha(\alpha q^k) \alpha(b) \]

and \( \alpha(0) = 1 \) where \( q \) is a given integer larger than 1. The typical example of q-multiplicative sequences is the sum of digits \( s_q(n) \): a positive integer \( n \) is written q-adically

\[ n = \sum_{k=0}^{\infty} a_k \cdot q^k \]

then

- 5 -
\[ s_q(n) = \sum_{k=0}^{\infty} a_k. \]

From the Bochner-Herglotz representation theorem, there exists a bounded positive measure \( \Lambda_\alpha \) such that

\[ \gamma_\alpha(h) = \int_{\mathbb{R}/\mathbb{Z}} \exp(2i\pi hx) \, d\Lambda_\alpha(x). \]

This measure \( \Lambda_\alpha \) is called spectral measure of a given sequence \( \alpha \), and we can describe certain properties of the sequence \( \alpha \) by means of its spectral measure (20).

If a q-multiplicative sequence is almost periodic, then its spectral measure is atomic.

If a q-multiplicative sequence is pseudo-random, then its spectral measure is singular.

Back to the sum of digits. Let us define

\[ \xi(n) = \exp(2i\pi cs_q(n)), \]

where \( (q-1)c \notin \mathbb{Z} \). The spectral measure \( \Lambda_\xi \) is continuous but singular with respect to Lebesgue measure. Let \( p \) and \( q \) be two relatively prime integers larger than 2, and

\[ \alpha(n) = \exp(2i\pi as_p(n)), \beta(n) = \exp(2i\pi bs_q(n)), \]

where \( a \) and \( b \) are integers satisfying

\[ (p-1)a \notin \mathbb{Z}, (q-1)b \notin \mathbb{Z}. \]

Then \( \Lambda_\alpha \) and \( \Lambda_\beta \) are singular to each other, proved by Kamae.
This is a kind of reply to the Steinhaus problem 144 stated in my report "normal numbers and dimension".

Finally I remark that a series of papers on spectral measures of arithmetical sequences have been produced by Coquet.

LIST OF PUBLICATIONS OF MICHEL MENDES FRANCE


    (1964).

(3) Dimension de Hausdorff. Application aux nombres normaux,

(4) A set of nonnormal numbers, Pacific J.Math.,1165-1170
    (1965).

(5) Nombres normaux. Applications aux fonctions pseudo-

(6) Deux remarques concernant l'équirépartition des suites, Acta
    Arith.,14,163-167(1968).

(7) Nombres transcendants et ensembles normaux, Acta Arith.,15,
    189-192(1969). (Séminaire Delange-Pisot-Poitou:1967/68,
    Théorie des Nombres,Exp.16.)

(8) La réunion des ensembles normaux, J.Number Th.,2,345-351

(9) DRESS,F. et MENDES FRANCE,M.: Caractérisation des ensembles
dans Z, Acta Arith.,17,115-120(1970). (Séminaire Delange-
Pisot-Poitou:1968/69,Théorie des nombres,Exp.17. Erratum:
Séminaire Delange-Pisot-Poitou:1969/70,Théorie des nombres,
Exp.24.)

(11) Suites et sous-suites équiréparties modulo 1, Journées Arithmétiques Francaises(Univ.Provence,1971).


(17) A characterization of Pisot numbers, Mathematika(1975).

(18) Hasard et déterministe, Seminaire Théorie des Nombres, Bordeaux:1975/76.


(22) Apéry,irrationalité de ζ(3), Les Recherches(1978).
(23) Algebraic numbers and automata theory, Proc. Queen's Univ.,

(24) CUSICK, T.W., and MENDES FRANCE, M.: The Lagrange spectrum of a

(25) Nombres algébriques et théorie des automates, l'Enseignement

(26) CHRISTOL, G., KAMAET, T., MENDES FRANCE, M., et RAUZY, G.: Suites
algébriques, automates et substitutions, Bull. Soc. Math. France,

(27) Principes de la symétrie perturbée, Séminaire Delange-Pisot-
Poitou: 1979/80, Théorie des nombres, Exp. 15, Birkhäuser, Boston-

(28) de l'arbre de Leonardo da Vinci à la théorie de la dimension

(29) MENDES FRANCE, M. et TANENBAUM, G.: Dimension des courves
planes, papiers pliés et suites de Rudin-Shapiro, Bull. Soc.

(30) MENDES FRANCE, M. and van der POORTEN, A.J.: Arithmetic and
analytic properties of paper folding sequences, Bull.

(31) DEKKING, F.M., and MENDES FRANCE, M.: Uniform distribution
modulo one: a geometrical viewpoint, J. reine und Angew. Math.,

(32) Paper folding, space filling curves and Rudin-Shapiro

(33) BLANCHARD, A., et MENDES FRANCE, M.: Symétrie et transcendence,

(34) DEKKING, F.M., MENDES FRANCE, M. and van der POORTEN, A.J.: