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SPECTRAL THEORY ON UNIFORMLY DISTRIBUTED SEQUENCES

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1. INTRODUCTION.

We shall consider sequences of real numbers or of complex numbers. The difference between simply normality and normality is to consider the frequency of any kind of block B_k . For the sequence of real numbers modulo 1 the corresponding relation is like between uniform distributed sequences and completely uniformly distributed sequences.

Theory of the uniform distribution of sequences is a fruitful branch in mathematics where number theory, probability theory and harmonic analysis meet with. Taking a normal number x to base q , then the sequence $(q^n x)$ is uniformly distributed modulo 1 . Hence we start from a modification of uniformly distributed sequences modulo 1 . Then we discuss the spectrum and the spectral measure of a given sequence. These notions are from Fourier analysis and shall explain certain properties of the sequence considered.

2. NORMAL SETS.

Let $\Lambda = (\lambda_n)$ be an infinite sequence of real numbers. A real number x is called Λ-normal if the sequence $x\Lambda = (x\lambda_n)$ is uniformly distributed modulo 1. Taking $\Lambda = (q^n)$, then a Λ-normal number coincides with a usual normal number to base q. The set of Λ-normal numbers is denoted by $B(\Lambda)$. A set E of real numbers is called normal if there exists a sequence Λ such that $E = B(\Lambda)$.

Mendès France and his colleagues collaborated on this notion (7), (8), (9) and (10) just after his arrival to Bordeaux University. Examples of normal sets are the followings:

i) the complement of any real algebraic field of finite degree;

ii) $m\mathbb{Z}^*$ with positive integer m;

iii) $\mathbb{Q}^*$.

This series of papers ended with Gérard Rauzy's paper which gave a complete characterization of the normal set:

THEOREM (RAUZY). A set B is normal if and only if

i) $0 \notin B$;

ii) for every positive integer q, $qB \subset B$;

iii) there exist continuous real functions $f_1, f_2, \ldots$ such that $\lim_{n\to\infty} f_n(x) = 0$ exactly for those x's which are elements of B .

The remainning unsolved (?) problems on normal sets are as follows:

1) Characterize the number μ such that $B + \mu = B$ implies the normality of B . (Every positive integer satisfies this property.)

2) A finite union of normal sets is normal or not.

In early 70's Mendès France considered the "depth" of a rational number using continued fraction expansions (13) and (14) and mentioned a relation to the formal language theory (16) and corresponding results on real algebraic numbers of second degree (remind Lagrange's theorem on periodic continued fractions) were proved by Henri Cohen and also by Peysant Leroux. Another related his paper to continued fractions is (24).

3. SPECTRUMS AND SPECTRAL MEASURES.

In analyzing a sequence of statistical data, the autocorrelation functions are the main powerful tool. This idea is back to Nobert Wiener in 1927 for time series analysis. The progress of Fourier analysis allows us to apply this idea on arithmetical sequences.

From now on we considered arithmetical sequences $\alpha = (\alpha(n))$ of complex numbers. The correlation function of with lag h is defined by

$$\gamma_\alpha(h) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \alpha(n)\overline{\alpha(n+h)} ,$$

where $\overline{z}$ is the complex conjugate of z . M.Bass defined a pseudo-random sequence $\alpha = (\alpha(n))$ satisfying following three conditions:

i) For each h , $\gamma_\alpha(h)$ is determined;

ii) $\gamma_\alpha(0) \neq 0$;

iii) $\lim_{h\to\infty} \gamma_\alpha(h) = 0$.

Mendès France proved the next theorem in his thesis (5):

THEOREM (MENDES FRANCE). If a real number x is normal to tne base r , then the sequence

$$n \longmapsto r_{|n|}(x) = \exp\left(\frac{2i\pi}{r} [x \cdot r^{|n|}]\right)$$

is pseudo-random and its correlation function is

$$\gamma(h) = \begin{cases} 1 & h = 0 \\ 0 & \text{otherwise} \end{cases} .$$

A normal number is a model of a random sequence, hence this theorem seems quite natural.

By replacing iii) by

$$\lim_{N\to\infty} \frac{1}{N} \sum_{h=1}^{N} |\gamma(h)|^2 = 0,$$

we call $\alpha = (\alpha(n))$ is pseudo-random in the sense of Bertrandias. From now on we used "pseudo-random" in the sense of Bertrandias.

Let us introduce another notion: spectrum of a sequence $\alpha = (\alpha(n))$:

$$sp(\alpha) = \{ \lambda \in [0,1); \limsup_{N\to\infty} \frac{1}{N} |\sum_{k=1}^{N} \alpha(k)\exp(-2i\pi\lambda k)| < \infty \}$$

Van der Corput difference theorem said that pseudo-random sequence has empty spectrum. The converse is not in general true. But in certain class of sequences, these two notions are equivalent. Indeed, for any q-multiplicative sequence empty spectrum implies pseudo-random (19). An arithmetical function α is called q-multiplicative if

$$\alpha(aq^k + b) = \alpha(aq^k)\, \alpha(b)$$

and $\alpha(0) = 1$ where q is a given integer larger than 1. The typical example of q-multiplicative sequences is the sum of digits $s_q(n)$: a positive integer n is written q-adically

$$n = \sum_{k=0} a_k \cdot q^k$$

then

$$s_q(n) = \sum_{k=0} a_k .$$

From the Bochner-Herglotz representation theorem, there exists a bounded positive measure Λ_α such that

$$\gamma_\alpha(h) = \int_{\mathbb{R}/\mathbb{Z}} \exp(2i\pi hx)\, d\Lambda_\alpha(x) .$$

This measure Λ_α is called spectral measure of a given sequence α , and we can describe certain properties of the sequence α by means of its spectral measure (20).

If a q-multiplicative sequence is almost periodic, then its spectral measure is atomic.

If a q-multiplicative sequence is pseudo-random, then its spectral measure is singular.

Back to the sum of digits. Let us define

$$\xi(n) = \exp(\ 2i\pi c s_q(n)\ ),$$

where $(\ q - 1\ )c \notin Z$. The spectral measure Λ_ξ is continuous but singular with respect to Lebesgue measure. Let p and q be two relatively prime integers larger than 2 , and

$$\alpha(n) = \exp(\ 2i\pi a s_p(n)\ ),\ \beta(n) = \exp(\ 2i\pi b s_q(n)\ ),$$

where a and b are integers satisfying

$$(\ p - 1)a \notin Z,\ (\ q - 1\ )b \notin Z.$$

Then Λ_α and Λ_β are singular to each other, proved by Kamae.

This is a kind of reply to the Steinhaus problem 144 stated in my report "normal numbers and dimension".

Finally I remark that a series of papers on spectral measures of arithmetical sequences have been produced by Coquet.

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