Some Applications of Fourier Analysis

to Uniform Distribution Mod 1.

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We say that a sequence \((\lambda_n)_1^n\) of real numbers is uniformly distributed mod 1 (u.d. mod 1 ) if we have for any \(0 \leq \alpha < \beta \leq 1\)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1 = \beta - \alpha,
\]

\[
\alpha \leq \{\lambda_k\} \leq \beta
\]

where \(\{\lambda_k\}\) denotes the fractional part of \(\lambda_k\). Then Weyl proved that \((\lambda_n)_1^n\) is u.d. mod 1 iff

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i h \lambda_k} = 0,
\]

for all fixed natural numbers \(h\). Among other important results, he gave the following "metric result".

Theorem A. If \((a_n)_1^n\) is a sequence of distinct natural numbers, then \((xa_n)_1^n\) is u.d. mod 1 for almost all fixed \(x\).

In fact, behind his proof of this theorem, a much more general principle was laid. For instance, the following generalization is possible [3].
Theorem B. If \( (\lambda_n) \) is a sequence of real numbers such that
\[
\inf_{n \neq m} |\lambda_n - \lambda_m| > 0,
\]
then the sequence \( (x\lambda_n) \) is u.d. mod 1 for almost all real numbers \( x \).

The main object of this article is to indicate another approach so that we can improve these results to some extent.

Let
\[
(1) \quad \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]
be the Fourier series of \( f \in L^1 \).

In 1920 around, Russian mathematician Lusin made the conjecture that for all \( f \in L^2 \), (1) converges for almost all \( x \). About ten years later, Kolmogorov constructed an \( f \in L^1 \) such that (1) diverges for all \( x \). However, in 1966 L. Carleson [1] has proved the Lusin conjecture affirmatively. A version of his theorem is

Theorem C. If \( \sum_{n=1}^{\infty} (a_n^2 + b_n^2) < \infty \), then (1) converges for almost all \( x \).

It will be worth stating here an open problem due to Orlicz [4]: Give an example of (1) divergent for all \( x \) and such taht
\[
\sum_{n=1}^{\infty} (a_n^{2+\varepsilon} + b_n^{2+\varepsilon}) < \infty
\]
for every \( \varepsilon > 0 \).

We may now apply Theorem C to the Fourier integral
\[
\int_0^{+\infty} f(t) e^{i\xi t} dt
\]
and obtain the following
Theorem 1. If \( \lambda_{k+1} - \lambda_k \geq \kappa > 0 \) \((k=1,2,\ldots)\) and
\[
\sum_{k \neq 1} |a_k|^2 < \infty,
\]
then
\[
\sum_{k \neq 1} \frac{i\lambda_k x}{a_k e^{\lambda_k x}}
\]
converges for almost all \( x \).

Corollary 1. Under the assumption of Theorem 1,
\[
\sum_{k=1}^{\infty} \frac{i\lambda_k x}{1 + a_k e^{\lambda_k x}}
\]
converges for almost all \( x \).

Corollary 2. If \( \lambda_{k+1} - \lambda_k \geq \kappa > 0 \) \((k=1,2,\ldots)\), then
\[
\sum_{k=1}^{N} \frac{i\lambda_k x h}{e^{\lambda_k x}} = 0 \left( \sqrt{N \log N} \right)^{(\log \log N)^{\delta}}
\]
for almost all \( x \), where \( h \in \mathbb{N} \) is fixed and \( \delta > 1/2 \) an arbitrary constant.

More generally, we can in fact prove

Theorem 2. If
\[
\lambda_{k+1} - \lambda_k \geq \mu_k > 0, \quad (k=1,2,\ldots)
\]
\[
\lambda_k \to \infty \quad (k \to \infty),
\]
and for almost all \( x \)
\[
\sum_{k \neq 1} \frac{\mu_k}{\sin^2(\mu_k x/2)} |a_k|^2 < \infty,
\]
then (2) converges for almost all \( x \).
Corollary 3. The Dirichlet series
\[ \sum_{n \leq 1} \frac{c_n}{n^s} \quad (s = \sigma + it) \]
converges on \( \sigma = 1/2 \) for almost all \( t \) if
\[ \sum_{n \leq 1} |c_n|^2 < \infty. \]

From Theorem 2 we obtain the following

Theorem 3. If (3) holds and for almost all \( x \)
\[ \sum_{k \leq 1} \frac{\mu_k}{k^2 \sin^2(\mu_k x/2)} < \infty, \]
then the sequence \((x\lambda_k)_1^\infty\) is u.d. mod 1 for almost all \( x \).

If we denote by \( p_n \) the \( n \)th prime \((p_1=2)\), then for sufficiently large \( n \) we have
\[ (\log p_{n+1})^\delta - (\log p_n)^\delta \gg \frac{(\log n)^{\delta - 2}}{n}, \quad (\delta > 0) \]
by the prime number theorem and Hoheisel's theorem. Thus the following interesting result is immediately deduced from Theorem 3.

Corollary 4. The sequence \((x(\log p_n)^\delta)_1^\infty\) is u.d. mod 1 for almost all \( x \), provided \( \delta > 3 \).

Our method cannot afford anything if \( 0 < \delta \leq 3 \), and we are tempted to conjecture that Corollary 4 is still true if \( \delta > 1 \). We can prove, however, that \((x(\log p_n)^\delta)_1^\infty\) is \((M, (\log n)^{\delta-1}/n)\) - u.d. mod 1 for almost all \( x \) if \( \delta > 1 \) (cf. Corollary 5 below). A sequence \((\lambda_n)_1^\infty\) of real numbers is said to be \((M, \alpha_n)\) - u.d. mod 1 if there exists a sequence of positive numbers such that
\[ \alpha_1 > \alpha_2 > \cdots > \alpha_n > \cdots \]

\[ \sum_{n=1}^{\infty} \alpha_n = \infty \]

and

\[ \frac{N}{\sum_{n=1}^{N} \alpha_n} e^{2\pi i h \lambda_n} = 0 \quad (\sum_{n=1}^{\infty} \alpha_n) \]

for any fixed \( h \in \mathbb{N} \).

It is easy to see that any u.d. sequence is \((M, \alpha_n) - u.d.\) for any \( \alpha_n \).

By a similar argument due to Cossar [2], we obtain

Theorem 4. If (3) holds and for almost all \( x \)

\[ \sum_{k=1}^{\infty} \left| \frac{c_k}{\lambda_k^p} \right| \cdot \frac{\mu_k}{|\sin (\mu_k x/2)|^p} < \infty , \]

for some \( 1 < p \leq 2 \), then

\[ \sum_{k=1}^{n} \frac{i \lambda_k \lambda_k}{c_k} e^{i \lambda_k x} = 0 \quad (\lambda_{n+1}) \]

for almost all \( x \).

Corollary 5. If \( \lambda_n \ll \sum_{k=1}^{n} c_k \) and

\[ \lambda_{k+1} - \lambda_k \gg 1/(\log k)^{\alpha} \]

for some \( 0 \leq \alpha < 1 \), then \((x \lambda_k)^n\) is \((M, c_n) - u.d. \) mod 1 for almost all \( x \).

We remark that our argument breaks down if \( \alpha = 1 \). So it does not follow that \((x \log p_n)^n\) is \((M, 1/n) - u.d. \) mod 1 for almost all \( x \).
References


