<table>
<thead>
<tr>
<th>Title</th>
<th>On the Cowell-Numerov Type Difference Equation Generated by Finite Elements (Functional Equations and Their Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Ikeuchi, Masatoshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1983, 499: 135-142</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1983-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/103651">http://hdl.handle.net/2433/103651</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On the Cowell-Numerov Type Difference Equation
Generated by Finite Elements

Okayama University of Science
Masatoshi Ikeuchi 池内 雅紀

The Cowell-Numerov type difference operators are generated for
n-dimensional linear eigensystems by non-standard finite elements
and the convergence theorem for the eigenvalue is proved.
Numerical example is also shown.

1. Introduction

The paper is concerned with the linear difference equation
(linear eigensystem) in the form

(E) \[ A_h U_h = \lambda_h B_h U_h \quad \text{in } \Omega \subset \mathbb{R}^n \]

where \( A_h, B_h \) are bounded holomorphic for the parameter \( h \in \mathbb{R}_+ \), and
\( U_h \) is the solution associated with the eigenvalue \( \lambda_h \in \sigma_h(B_h^{-1}A_h) \).
Let (E) be the discrete approximation to the original differential
equation \( A u = \lambda B u \). Assume that \( A_h, B_h \) are second-order.
Then the best rational approximant to the characteristic solution
of the original equation gives rise to the Cowell-Numerov operator
in \( \Omega \subset \mathbb{R}^1 \) [Froberg 1970], Lambert 1979].

The objective is to form the Cowell-Numerov type (C.N.) operator
in \( \Omega \subset \mathbb{R}^n \) by the use of finite elements [Milne 1980] and establish
the convergence theorem for the eigenvalue \( \lambda_h \rightarrow \lambda \in \sigma(B^{-1}A) \).
Unfortunately, the standard finite elements [Strang & Fix 1973]
cannot generate the C.N. operators. Thus we start with
non-standard finite elements.

2. Preliminary

We prepare the notations:

\( \Omega \) the open and bounded polygon \( \subset \mathbb{R}^n \),
\[ \Delta, \] the Laplacian \( \partial^2/\partial x_i^2 \) (i=1,2,...,n),
\[ R_+ \] the real subset \([0, +\infty)\),
\[ H^m(\Omega) \] the m-th order Hilbert space with the inner product
\[ (u,v)_m,\Omega = \int_\Omega \sum_{k=1}^m |k| \langle u/\partial x_i^k, v/\partial x_i^k \rangle \, dx \]
and the semi-norm
\[ |v|_m,\Omega = \left\{ \int_\Omega (\partial^m v/\partial x_i^m)^2 \, dx \right\}^{1/2}, \]
\[ a(v,v) \] the (stiffness) energy inner product \( |v|_1,\Omega \),
\[ b(v,v) \] the (mass) energy inner product \( |v|_0,\Omega \).

Now we present the original eigenproblem
\[ (P1) \text{ find } (\lambda, u) \in R_+ \times H^2(\Omega), \text{ such that } \]
\[ -\Delta u = \lambda u \text{ in } \Omega \text{ for } u = 0 \text{ on } \partial \Omega. \]
It is known [Strang & Fix 1973, Ciarlet 1978] that (P1) becomes equivalent to
\[ (P2) \text{ find } (\lambda, u) \text{ in the admissible space } R_+ \times H^1_0(\Omega), \text{ such that } \]
\[ a(u,v) = \lambda b(u,v) \text{ for all } v \text{ in } H^1_0(\Omega). \]

3. Finite Element Spaces

For the formulation of non-standard finite elements, we define the finite-dimensional subspaces \( S_h \) and \( V_h^\alpha \), as follows.
\[ (T1) \text{ The trial space } \]
\[ S_h := \text{span}\{F_1, \ldots, F_N\} \subset H^1_0(\Omega) \]
in which for the piecewise linear function \( f_i(x_k) \) with compact support
\[ F_i(x) = \prod_{k=1}^n f_i(x_k), \text{ and } \dim(S_h) = N. \]
\[ (T2) \text{ The test space } \]
\[ V_h^\alpha := \text{span}\{W_1^\alpha, \ldots, W_N^\alpha\} \subset H^1_0(\Omega) \]
in which for the piecewise cubic function \( w_i^\alpha(x_k) \) with compact support
\[ W_i^\alpha(x) = \prod_{k=1}^n w_i^\alpha(x_k), \text{ the parameter } \alpha = (\alpha_1, \ldots, \alpha_n) \in R_+^n, \]
\[ \text{ and } \dim(V_h^\alpha) = N. \]
Here, we choose \( w_i^\alpha(x_k) \) in the form
\[ w^\alpha_i(x_k) = f_i(x_k) + \alpha_k g_i(x_k) \]

where \( g_i(x_k) \) is the cubic even function satisfying
\[
\int_{\Omega} g_i(x_k) \, dx_k = 0.
\]

Thus, using (T1) and (T2), our problem (P2) can be approximated as

\[(P3) \quad \text{find} \ (\lambda_h, u_h) \ \text{in the trial space} \ R_+ \times S_h, \ \text{such that} \]
\[
a(u_h, v_h) = \lambda_h b(u_h, v_h) \ \text{for} \ v_h \ \text{in the test space} \ V_h^\alpha.
\]

From (P3) we can derive the following statements:

(S1) (P3) is equivalent to the standard finite element formulation for (P1) if and only if the parameters \( \alpha_k = 0 \)

(k=1,2,...,n) are chosen.

(S2) For arbitrary parameters \( \alpha_k \) (k=1,2,...,n), the minimal subspace of \( V_h^\alpha \) is included in \( S_h \).

(Proof) We know from (T2) that
\[
\psi v_h = \sum_{i=1}^{N} c_i (f_i + \alpha_1 g_i) \in V_h^\alpha; \ \psi c_i \in R
\]
\[
= \sum_{i=1}^{N} c_i f_i \in S_h
\]

for \( n = 1 \).

4. Difference Equation

As the trial function \( u_h \in S_h \) we set
\[
u_h(x) = \sum_{i=1}^{N} U_{hi} F_i(x)
\]

where \( U_{hi} = u_h(x^i) \) is the nodal eigensolution at \( x^i = (x_1,\ldots,x_n) \).

Then the finite element solution \( (\lambda_h, U_h) \) to (P3) satisfies the difference equation

\[(D) \quad A^\alpha_h U_h = \lambda_h B^\alpha_h U_h \]

in which \( U_h = \{U_{h1},\ldots,U_{hN}\} \).

Let us show some examples for the second-order difference operators \( A_h \) and \( B_h \) in (D).

(Ex1) Case of \( n = 1 \):
\[ A^\alpha_h = -E_1 + 2 I - E_1^{-1} = -\delta_1^2, \]
\[ B^\alpha_h = (h^2/6)[(1-\alpha_1) E_1 + 2(2+\alpha_1) I + (1-\alpha_1) E_1^{-1}] \]
\[ = (h^2/6)[(1-\alpha_1) \delta_1^2 + 6I] \]
\[ - 3 \]
where $E_k$ is the shift operator to $x_k$ direction and $\delta_k$ is the central difference operator.

From (EX1) we have readily the following statements:

(S3) For an arbitrary parameter $\alpha_1 \in \mathbb{R}^+$, $A_h$ and $B_h$ satisfy the consistency conditions [Lambert 1979].

(S4) For the parameter $\alpha_1 = 1/2$,

$$B_h^{(1/2)} = (h^2/12)[\delta_1^2 + 12I]$$

which is the Cowell-Numerov operator [Henrici 1962, Froberg 1970, Lambert 1979].

(S5) For the parameter $\alpha_1 = 0$,

$$B_h^{(0)} = (h^2/6)[\delta_1^2 + 6I]$$

which is the standard finite element mass operator.

Therefore we call the Cowell-Numerov type (C.N.) operators both $A_h^\alpha$ and $B_h^\alpha$ with $\alpha = (1/2)$ in the paper.

(EX2) Case of $n = 2$:

$$A_h^\alpha = \left[ 4(4+\alpha_1+\alpha_2)I - 2(1+\alpha_1+\alpha_2) (E_1 + E_1^{-1} + E_2 + E_2^{-1}) - (2-\alpha_1-\alpha_2) (E_1E_2 + E_1E_2^{-1} + E_1^{-1}E_2 + E_1^{-1}E_2^{-1}) \right]/6,$$

$$B_h^\alpha = (h^2/36) \left[ 4(2+\alpha_1)(2+\alpha_2)I + 2(2+\alpha_1)(1-\alpha_2)(E_1 + E_1^{-1} + E_2 + E_2^{-1}) + (1-\alpha_1)(1-\alpha_2)(E_1E_2 + E_1E_2^{-1} + E_1^{-1}E_2 + E_1^{-1}E_2^{-1}) \right]$$

\[ \begin{array}{ccc}
-1 & -4 & -1 \\
-4 & 20 & -4 \\
-1 & -4 & -1
\end{array} \]  \hspace{1cm} \begin{array}{ccc}
1 & 10 & 1 \\
10 & 100 & 10 \\
1 & 10 & 1
\end{array} \]

c.d. = 6  \hspace{1cm} \text{c.d.} = 144h^2

\textit{Fig. 1A Cowell-Numerov type stencils for } n = 2.
(EX21) The whole C.N. stencils for \( n=2 \) are illustrated in Fig.1A.

(EX31) The C.N. stencils for \( n=3 \) are illustrated in Fig.1B.

![Diagram showing C.N. stencils for \( n=2 \) and \( n=3 \)]

\[
\begin{align*}
A_h^{(1/2)} & : & \begin{array}{c}
-1 \\
-3 \\
-5 \\
-5 \\
25 \\
\end{array} & \\
B_h^{(1/2)} & : & \begin{array}{c}
1 \\
5 \\
25 \\
25 \\
125 \\
\end{array}
\end{align*}
\]

c.d. = 48 \hspace{1cm} \text{c.d.} = 1728h^2

\textit{Fig.1B Cowell-Numerov type stencils for } n = 3.

5. Error Analysis

We discuss on the error analysis of the eigenvalue \( \lambda_h \) in (D).

For the characteristic solutions to the original equation in (P1), we meet with the approximate problem

(A1) \[ \exp[\pm i\sqrt{\lambda_h}] = R_k + i(1-R_k^2)^{1/2} + \epsilon_k/2 \]

where \( i^2 = -1 \), \( \epsilon_k \) is the residual term and

\[ R_k(s) = [1-(s^2/6)(2+\alpha_k)]/[1+(s^2/6)(1-\alpha_k)] \] for \( s=\sqrt{\lambda_h} h \).

For simplicity, instead of (A1), we can consider the rational approximate problem in the form

(A2) \[ \cos(\sqrt{\lambda_h}) = R_k(\sqrt{\lambda_h} h) + \epsilon_k \] for \( k=1, \ldots, n \).

From the Padé approximate theory [Cheney 1966, Brezinski 1980], we have the following statements:

(S6) The parameter \( \alpha_k = 1 \) gives the \((2/0)\)-Padé approximant to \( \cos(\sqrt{\lambda h}) \) in (A2).

(S7) The parameter \( \alpha_k = 1/2 \) gives the \((2/2)\)-Padé approximant to \( \cos(\sqrt{\lambda h}) \) in (A2).

In Table 1, we give the \((l/m)\)-Padé approximants to the \( R_k(s) \).
Note that the standard finite element solution derivates from the Padé table, therefore it cannot give rise to the best rational approximation.

<table>
<thead>
<tr>
<th>$z=0$</th>
<th>$z=2$</th>
<th>$z=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m=0$</td>
<td>$1/1$</td>
<td>$(1-s^2/2)/1$</td>
</tr>
<tr>
<td></td>
<td>(meaningless)</td>
<td>$(1-s^2/3)/(1+s^2/6)$*</td>
</tr>
<tr>
<td>$m=2$</td>
<td>$(1-5s^2/12)/(1+s^2/12)$</td>
<td>nonlinear</td>
</tr>
<tr>
<td></td>
<td>$\uparrow$ linear</td>
<td>$\uparrow$ eigensystem</td>
</tr>
<tr>
<td>$m=4$</td>
<td></td>
<td>$(1-115s^2/252+313s^4/15120)/$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1+11s^2/252+13s^4/15120)$</td>
</tr>
</tbody>
</table>

Note: $s = \sqrt[\lambda_h]{h}$,

*standard linear finite elements.

From the statement (S6) and the result in Table 1, we have the following convergence theorem:

(Th1) Let $\lambda$ and $\lambda_h$ be sufficiently small numbers. There exists a positive constant $M_1$ such that

$$|\sqrt[\lambda_h]{s} - \sqrt[\lambda]{s}| \leq M_1 \lambda^{3/2} h^2$$

for the parameters $\alpha_k = 1$ (k=1,...,n).

From the statement (S7) we have the following convergence theorem with respect to the C.N. operators:

(Th2) Let $\lambda$ and $\lambda_h$ be sufficiently small numbers. There exists a positive constant $M_2$ such that

$$|\sqrt[\lambda_h]{s} - \sqrt[\lambda]{s}| \leq M_2 \lambda^{5/2} h^4$$

for the parameters $\alpha_k = 1/2$ (k=1,...,n).

(Proof) We write

$$R_k(s) = (1-5s^2/12)/(1+s^2/12)$$

for $\alpha_k = 1/2$, in which $s = \sqrt[\lambda_h]{h}$. By the total differentials we have

$$\Delta R_k = -[s/(1+s^2/12)^2] \Delta s = -\varepsilon_k$$
where \( \Delta s = (\sqrt{\lambda_h} - \sqrt{\lambda})h \). Thus we obtain

\[
\Delta s \approx (1 + s^2/12) \epsilon_k \lesssim M_2 s^6
\]

for some positive constant \( M_2 \).

6. Numerical Example

We examine in numerical experiments the validity of the C.N. operators for \( n = 3 \).

Fig. 2 shows the convergence characteristics for the ordering number of \( \lambda_h \) and the parameter (or element size) \( h \).

---

**Fig. 2 Convergence characteristics.**
It is directly seen from the result in Fig.2 that (Th2) is valid and the C.N. operators are more useful than the standard finite element operators.

7. Conclusion

The main results in the paper are summarized, as follows:

1) The Cowell-Numerov type (C.N.) operators are generated in \( \Omega \subset \mathbb{R}^n \) by the non-standard finite elements.

2) The trial space \( S_h (\subset H^1_0(\Omega) ) \) and the test space \( V^\alpha_h \subset S_h \) are formed for the non-standard finite elements.

3) The C.N. operators give rise to the best rational approximant to the characteristic solution.

4) The convergence theorem for the eigenvalue \( \lambda_h \) associated with the C.N. operators is established by the Pade approximate theory.

It can be concluded that the C.N. operators are efficient for linear eigensystems, and that the non-standard finite elements are more extensive.

References:


