

Mean Value and Difference Type Functional Equations

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§ 1. Introduction.

$x, y, x_i, y_i \in \mathbb{R}^n$ or a linear space X

$t, r, t_i, r_i \in \mathbb{R}$

$f: \mathbb{R}^n \rightarrow \mathbb{R}$ or $f: X \rightarrow Y$ (another linear space)

Type I

$$f(x) = F\{f(x+t_1y), \dots, f(x+t_ky), x, y\}$$

$\forall x, y \in \mathbb{R}^n$ (or some given subsets of \mathbb{R}^n),

fixed $t_i \in \mathbb{R}$

Type II

$$f(x) = F\{f(x+ty_1), \dots, f(x+ty_k), x, t\}$$

$\forall x \in \mathbb{R}^n, t \in \mathbb{R}$ (or subsets of \mathbb{R}^n and \mathbb{R}),

fixed $y_i \in \mathbb{R}^n$

Note: $n=1$, Type I = Type II

Problem:

1. the general solution
2. a very weak regularity assumption
 \Rightarrow continuous, or C^∞
 very often that f is a polynomial on \mathbb{R}^n

Examples of Type I

The finite difference equations in \mathbb{R}^n

$$(1) \quad \Delta_y^{k+1} f(x) = g(x, y) \quad \forall x, y \in \mathbb{R}^n$$

Equations (1) include as particular cases:

(a) Cauchy's functional equation in \mathbb{R}^n

$$f(x+y) = f(x) + f(y) \iff \Delta_y^1 f(x) = f(y)$$

(b) Jensen's functional equation in \mathbb{R}^n

$$f\left(\frac{u+v}{2}\right) = \frac{1}{2} \{f(u) + f(v)\} \iff \Delta_y^2 f(x) = 0$$

(c) Quadratic functional equation in \mathbb{R}^n

$$f(u+v) + f(u-v) = 2f(u) + 2f(v)$$

$$\text{Set } u = x+y, \quad v = y$$

$$\iff \Delta_y^2 f(x) = 2f(y)$$

(d) $f(u+v) + f(u-v) = 2f(u) + 2f(v) + g(u, v)$

⋮

Example of Type II

The mean value equations in \mathbb{R}^n , for example,

$$(2) \quad \sum_{i=1}^k \lambda_i f(x + t y_i) = f(x), \quad \lambda_i \in \mathbb{R}, \quad \sum_{i=1}^k \lambda_i = 1$$

Here, the y_i are interpreted as the position vectors of the vertices of a k -gon in \mathbb{R}^n (regular or not), and the λ_i are weights at each vertex. Equation of type (2) then demand that the value $f(x)$ of f at the center x of the k -gon should equal the (weighted) mean value of f at the vertices of the k -gon.

Equations (2) include as particular cases:

(a) Jensen's functional equation

$$f\left(\frac{u+v}{2}\right) = \frac{1}{2} \{f(u) + f(v)\} \iff f(x+y) + f(x-y) = 2f(x)$$

(b) Kakutani-Nagumo-Wold's functional equation

$$f: \mathbb{C} \rightarrow \mathbb{R}, \quad \theta = \exp(2\pi i/n), \quad n \geq 3,$$

$$\sum_{\sigma=0}^{n-1} f(x + \theta^\sigma y) = n f(x)$$

⋮

§ 2. Linear Type I: Regular Solutions

Consider equations of the form

$$(3) \quad f_0(x) + \sum_{i=1}^k f_i(x+t_i y) = \phi(x, y) \quad \text{for fixed } 0 \neq t_i \in \mathbb{R}$$

where the equation is assumed to hold for:

$$(4) \quad \left\{ \begin{array}{l} \text{all } x \in D \subseteq \mathbb{R}^n, \text{ where } D \text{ is open and connected,} \\ \text{all } y \in E \subseteq \mathbb{R}^n, \text{ where } \mu(E) > 0, \\ \phi \text{ is a given mapping } \phi: D \times E \rightarrow \mathbb{R}. \\ f_0: D \rightarrow \mathbb{R} \text{ and } f_i: D + t_i E \rightarrow \mathbb{R} \text{ are unknown functions.} \end{array} \right.$$

Theorem 1 (J. H. B. Kemperman 1957). Let f_0 and \bar{f}_0 be any two solutions of (3), given (4), and assume that f_0 and \bar{f}_0 are bounded in absolute value on sets S_0, \bar{S}_0 where $S_0 \subseteq D, \bar{S}_0 \subseteq D, \mu(S_0) > 0, \mu(\bar{S}_0) > 0$. (Note: no assumptions concerning the remaining f_i and $f_i!$). Then:

- i) if, for each $y_0 \in E$, $\phi(x, y_0)$ is continuous, or a polynomial, or analytic in $x \in D$, then f_0 and \bar{f}_0 are also continuous, or polynomials, or analytic in D .
- ii) if, for each $y_0 \in E$, $\phi(x, y_0)$ is continuous for $x \in D$, then $f_0 - \bar{f}_0$ is a polynomial of degree at most $k-1$ on D .
- iii) if, for each $y_0 \in E$, $\phi(x, y_0)$ is a polynomial for $x \in D$ of degree $\leq P$, then f_0 (and also \bar{f}_0) are polynomials of degree $\leq k+P$ on D .

Examples of Theorem 1. In the following examples, it is only assumed that the solution is bounded in absolute value on a set S , $\mu(S) > 0$ for a set $S \subseteq \mathbb{R}^n$.

Example 1. Write Pexider's equation

$$f(x+y) = g(x) + h(y)$$

in the form $f_1(x+y) - f_0(x) = \phi(y)$, as an equation in \mathbb{R}^n . Then we have that $g(x)$ is a polynomial of degree ≤ 1 .

Example 2. Write the equation

$$f(x+y) + h(x-y) = 2g(x) + 2k(y)$$

in the form $f_2(x+y) + f_1(x-y) - 2f_0(x) = 2\phi(y)$, whence $g(x)$ is a polynomial of degree ≤ 2 .

Example 3. For the equation

$$\Delta_y^k f(x) = g(y),$$

f is a polynomial of degree $\leq k$.

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§ 3. Linear Type I: General Solutions

Let X and Y denote arbitrary linear spaces over the reals \mathbb{R} . Consider the general finite difference functional equation

$$(5) \quad \Delta_y^{k+1} f(x) = 0 \quad \forall x, y \in X$$

\Leftrightarrow

$$\sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^{k+1-i} f(x+iy) = 0 \quad \forall x, y \in X$$

where $f: X \rightarrow Y$. The most extensive treatment of the general solution of (5) was given by S. Mazur and W. Orlicz 1934 (among others), and of

$$\Delta_y^{k+1} f(x) = \phi(y) \quad \forall x, y \in X$$

by G. van der Lijn 1945 (among others).

In order to describe the most general solution of these equations, we introduce the following notation:

$A_p: X^p \rightarrow Y$ denotes a symmetric multi-additive function, that is, $A_p(x_1, \dots, x_p) = A_p(x_{i_1}, \dots, x_{i_p})$ for all permutations (i_1, \dots, i_p) of $(1, \dots, p)$ and A_p satisfies Cauchy's functional equation in each X_α .

$A^p: X \rightarrow Y$ is defined by $A^p(x) = A_p(x, x, \dots, x)$ for all $x \in X$.

A function of the form A^P (that is, one for which such an A_p exists) is called rational homogeneous of order p . We take $A_0 = A^0$ to be the constant functions, rational homogeneous of order 0.

The results of Mazur, Orlicz, and van der Lijn include

Theorem 2. A necessary and sufficient condition that f be rational homogeneous of order p is that f satisfy

$$\Delta_y^p f(x) = p! f(y)$$

for all $x, y \in X$.

Example of Theorem 2. The well-known quadratic equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

may be written, as in (c),

$$\Delta_y^2 f(x) = 2f(y).$$

Hence the general solution is $f(x) = A_2(x, x)$ where $A_2(x, y)$ is an arbitrary symmetric and bi-additive function on $X \times X \rightarrow Y$.

Theorem 3. A necessary and sufficient condition that

$$\Delta_y^p f(x) = \phi(y)$$

have a solution is that $\phi(y) = A^p(y)$.

Theorem 4. A necessary and sufficient condition that

$$(5) \quad \Delta_y^{k+1} f(x) = 0 \quad \forall x, y \in X$$

is that $f(x) = \sum_{p=0}^k A^p(x).$

Let G and H be additive Abelian groups. Let S be any field and G, H be a unital S -modules. Let $f: G \rightarrow H$ satisfy the equation

$$(6) \quad \sum_{i=0}^n \gamma_i f(x + \alpha_i y) = 0 \quad \forall x, y \in G,$$

where $n > 2$ is a given integer, $\gamma_i \neq 0$, $\alpha_i \neq 0$ ($= \alpha_0$) for $i = 0, 1, \dots, n$ are fixed elements in S and $\alpha_j \neq \alpha_k$ for $j \neq k$. Equation (6) is a generalization of (5). More generally we have

Theorem 5 (S. Haruki 1981). Let $f_i: G \rightarrow H$ for $i = 0, 1, \dots, n$ satisfy the equation

$$(7) \quad \sum_{i=0}^n f_i(x + \alpha_i y) = 0 \quad \forall x, y \in G,$$

where $\alpha_i \neq 0$ for $i = 0, 1, \dots, n$ are fixed elements in S and $\alpha_j \neq \alpha_k$ for $j \neq k$. Then equation (7) implies

$$(8) \quad \Delta_u^n f_i(x) = 0$$

for each $i = 0, 1, \dots, n$ and for all $x, u \in G$.

§4. Linear Type II: General and Regular Solutions

Consider equations of the form

$$(9) \quad \sum_{i \in I} \lambda_i f(x + t y_i) = 0$$

where $\lambda_i, t \in \mathbb{R}$, the λ_i fixed and $\sum_{i \in I} \lambda_i = 0$, and where $x, y_i \in \mathbb{R}^n$ for fixed y_i . We use I to denote the set $\{1, 2, \dots, k\}$. The equation (9) is to hold for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ such that $x + t y_i \in D$ for all $i \in I$, and $D \subseteq \mathbb{R}^n$ is some (connected) domain.

Theorem 6 (M. A. McKiernan 1970). If $\sum_{i \in J} \lambda_i \neq 0$ for all subsets $J \subset I$ (proper subsets) then (9) implies that on D ,

$$\Delta_{t y_i}^p f(x) = 0$$

for all $i \in I$, and all $p > k(k-1)/2$.

Hence, if $\sum_{i \in J} \lambda_i \neq 0$ for all $J \subset I$, then the type II, in \mathbb{R}^n , is related to the type I in \mathbb{R}^n . In particular it easily follows that

Theorem 7 (M. A. McKiernan 1970). If further the y_i span \mathbb{R}^n , and if f is bounded in absolute value on a set $S \subseteq D$, $\mu(S) > 0$, then f is C^∞ on D , and hence a polynomial on D .