Actions of symplectic groups on a product of projective spaces

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0. Introduction

In previous papers [6],[7], smooth actions of special unitary (resp. symplectic) groups on a product of complex (resp. quaternion) projective spaces have been studied. Here we shall study smooth actions of symplectic group $\text{Sp}(n)$ on certain product manifolds and we shall prove the following.

Theorem. Let $X$ be a closed orientable manifold on which $\text{Sp}(n)$ acts smoothly and non-trivially. Suppose $n \geq 7$.

(i) Suppose $X \cong P_a(C) \times P_b(C), \ 1 \leq b \leq a < 2n$ and $a+b \leq 4n-3$. Then $a = 2n-1$ and $X$ is equivariantly diffeomorphic to $P_{2n-1}(C) \times Y_0$, where $Y_0$ is a closed orientable manifold such that $Y_0 \sim P_b(C)$ and $\text{Sp}(n)$ acts naturally on $P_{2n-1}(C)$ and trivially on $Y_0$.

(ii) Suppose $X \cong P_a(H) \times P_b(C), \ 1 \leq a \leq n-1, \ 1 \leq b \leq 2n-1$, and $2a+b \leq 4n-4$. Then there are three cases:

(a) $a = n-1$ and $X$ is equivariantly diffeomorphic to $P_{n-1}(H) \times Y_1$, where $Y_1$ is a closed orientable manifold such that $Y_1 \sim P_b(C)$ and $\text{Sp}(n)$ acts naturally on $P_{n-1}(H)$ and trivially on $Y_1$.

(b) $b = 2n-1$ and $X$ is equivariantly diffeomorphic to $P_{2n-1}(C) \times Y_2$, where $Y_2$ is a closed orientable manifold such that $Y_2 \sim P_a(H)$ and $\text{Sp}(n)$ acts naturally on $P_{2n-1}(C)$ and
trivially on $Y_2$.

(c) $b = 2n-1$ and $X$ is equivariantly diffeomorphic to $(S^{4n-1} \times Y_3)/\text{Sp}(1)$, where $Y_3$ is a closed orientable $\text{Sp}(1)$ manifold such that $Y_3 \sim S^2 \times P_a(H)$, $\text{Sp}(1)$ acts as right scalar multiplication on $S^{4n-1}$, the unit sphere of $H^n$, and $\text{Sp}(n)$ acts naturally on $S^{4n-1}$ and trivially on $Y_3$. In addition, $F \sim S^0 \times P_a(0)$ and the induced homomorphism $i^*: H^2(Y_3) \to H^2(F)$ is trivial, where $F$ denotes the fixed point set of the restricted $U(1)$ action on $Y_3$. Conversely, if $Y_3$ satisfies the above conditions, then $(S^{4n-1} \times Y_3)/\text{Sp}(1) \sim P_{2n-1}(C) \times P_a(H)$ for $1 \leq a \leq n-2$.

Throughout this paper, let $H^*(\ )$ denote the singular cohomology theory with rational coefficients. By $X_1 \sim X_2$ we mean $H^*(X_1) \cong H^*(X_2)$ as graded algebras. Denote by $P_n(C)$ and $P_n(H)$ the complex and quaternionic projective $n$-spaces, respectively.

1. Preliminary results

First we prepare the following two lemmas which are proved by a standard method (cf. [2], [3], [5]).

Lemma 1.1. Suppose $n \geq 7$. Let $G$ be a closed connected proper subgroup of $\text{Sp}(n)$ such that $\dim \text{Sp}(n)/G < 8n$. Then $G$ coincides with $\text{Sp}(n-i) \times K$ ($i = 1, 2, 3$) up to an inner automorphism of $\text{Sp}(n)$, where $K$ is a closed connected subgroup of $\text{Sp}(i)$.
Lemma 1.2. Suppose $r \geq 5$ and $k < 8r$. Then an orthogonal non-trivial representation of $\text{Sp}(r)$ of degree $k$ is equivalent to $(\mathcal{V}_r)_{\mathbb{R}} \otimes \mathcal{E}^{k-4r}$. Here $(\mathcal{V}_r)_{\mathbb{R}} : \text{Sp}(r) \to O(4r)$ is the canonical inclusion, and $\mathcal{E}^t$ is the trivial representation of degree $t$.

In the following, let $X$ be a closed connected orientable manifold with a non-trivial smooth $\text{Sp}(n)$ action, and suppose $n \geq 7$ and $\dim X < 8n$. Put

$$F(i) = \left\{ x \in X : \text{Sp}(n-i) \subset \text{Sp}(n)_x \subset \text{Sp}(n) \times \text{Sp}(i) \right\},$$

$$X(i) = \text{Sp}(n) F(i) = \left\{ g x : g \in \text{Sp}(n), x \in F(i) \right\}.$$ 

Here $\text{Sp}(n)_x$ denotes the isotropy group at $x$. Then, by Lemma 1.1, we obtain $X = X(0) \cup X(1) \cup X(2) \cup X(3)$. Moreover, from Lemma 1.2, we can show the following Propositions. The proofs are omitted.

Proposition 1.3. If $X(k)$ is non-empty, then $X(i)$ is empty for each $i \geq k+2$.

Proposition 1.4. Suppose $X = X(k) \cup X(k+1)$. If $X(k)$ and $X(k+1)$ are non-empty, then the codimension of each connected component of $F(k)$ in $X$ is equal to $4(k+1)(n-k)$.

Corollary 1.5. Suppose $X = X(2) \cup X(3)$. Then either $X(2)$ or $X(3)$ is empty.

Remark. $\dim \text{Sp}(n)/\text{Sp}(n-k) \times \text{Sp}(k) = 4k(n-k)$ and $\mathcal{K}(\text{Sp}(n)/\text{Sp}(n-k) \times \text{Sp}(k)) = nC_k$, where $\mathcal{K}(\ )$ denotes the Euler characteristic, and $nC_k$ denotes the binomial coefficient.
Remark. If \( \dim X < 4n \), then we see \( X = X(1) \). In addition, if \( H^{\text{odd}}(X) = 0 \), then \( X \) is equivariantly diffeomorphic to \( P_{n-1}(H), P_{n-1}(H) \times S^2 \) or \( P_{2n-1}(C) \), where \( \text{Sp}(n) \) acts naturally on \( P_{n-1}(H), P_{2n-1}(C) \) and trivially on \( S^2 \). So we assume \( \dim X \geq 4n \), in the following sections.

2. Cohomological aspects

Throughout this section, suppose that \( X \) is a closed orientable manifold with a non-trivial smooth \( \text{Sp}(n) \) action, \( n \geq 7 \) and \( X = X(0) \cup X(1) \).

Proposition 2.1. Suppose either \( X \sim P_a(C) \times P_b(C) \),
\[ 1 \leq b \leq a < 2n \leq a+b \leq 4n-3, \] or \( X \sim P_a(H) \times P_b(C), 1 \leq a \leq n-1, 1 \leq b \leq 2n-1, 2n \leq 2a+b \leq 4n-4. \] Then \( X(0) \) is empty.

(Proof) Suppose that \( X(0) \) is non-empty. Let \( U \) be an invariant closed tubular neighbourhood of \( X(0) \) in \( X \), and put \( E = X - \text{int } U \). Let \( i : E \to X \) be the inclusion. Then \( i^* : H^t(X) \to H^t(E) \) is an isomorphism for each \( t \leq 4n-2 \), because the codimension of each connected component of \( X(0) \) is \( 4n \) by Lemma 1.2. Put \( Y = E \cap F(1) \). Then \( Y \) is a connected compact orientable manifold with non-empty boundary \( \partial Y \), and \( \text{Sp}(1) \) acts naturally on \( Y \). There is a natural diffeomorphism \( E = (S^{4n-1} \times Y)/\text{Sp}(1) \). By the Gysin sequence of the principal \( \text{Sp}(1) \) bundle \( p : S^{4n-1} \times Y \to E \), we obtain an exact sequence:
\[
0 \to H^{2k-1}(S^{4n-1} \times Y) \to H^{2k-4}(E) \to H^{2k}(E) \to H^{2k}(S^{4n-1} \times Y) \to 0,
\]
where \( 2k = \dim Y = \dim X - (4n-4) \). Hence we obtain the rank \( H^{2k}(Y) \)
- rank $H^{2k-1}(Y) \geq 1$, by the cohomology ring structure of $X$.

Considering the homology exact sequence of the pair $(Y, \mathcal{X})$ and the Poincare-Lefschetz duality, we obtain

$$\text{rank } H_0(\mathcal{X}) \leq \text{rank } H_0(Y) + \text{rank } H^{2k-1}(Y) - \text{rank } H^{2k}(Y) \leq 0.$$ 

Therefore $\mathcal{X}$ is empty; this is a contradiction. q.e.d.

In the remaining of this section, we assume $X = X(1) = (S^{4n-1} \times F(1))/\text{Sp}(1)$, where $F(1)$ is a closed connected orientable manifold with a natural $\text{Sp}(1)$ action.

Here we describe certain situations which appear in the proofs of the following Propositions. Let $Y$ be a closed orientable $\text{Sp}(1)$ manifold such that $H^{\text{odd}}(Y) = 0$. Put $M = S^{4n-1} \times Y$, where $\text{Sp}(1)$ acts as right scalar multiplication on $S^{4n-1}$. Let $T$ be a closed toral subgroup of $\text{Sp}(1)$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
M/T & \xrightarrow{p_1} & M/\text{Sp}(1) \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
P_{2n-1}(C) & \xrightarrow{q} & P_{n-1}(H)
\end{array}
$$

where $\pi_1, \pi_2$ are projections of fibre bundles with $Y$ as the fibre, and $p_1, q$ are projections of 2-sphere bundles. Since $H^{\text{odd}}(Y) = 0$, we can apply the Leray-Hirsch theorem to the fibrations $\pi_1, \pi_2$. In particular, we see $H^{\text{odd}}(M/\text{Sp}(1)) = 0$.

By the Gysin sequence of the principal $\text{Sp}(1)$ bundle $p : M \to M/\text{Sp}(1)$, we obtain an exact sequence:

$$(A_1) \quad 0 \to H^{2i-1}(M) \to H^{2i-4}(M/\text{Sp}(1)) \xrightarrow{\mu} H^{2i}(M/\text{Sp}(1)) \to H^{2i}(M) \to 0$$

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for each \( i \), where \( \mu \) is the multiplication by \( e(p) \), the Euler class.

We regard \( S^\infty \) as the inductive limit of \( S^{4N-1} \) on which \( T \) acts naturally. Let \( F \) denote the fixed point set of the restricted \( T \) action on \( Y \). Consider the following commutative diagram:

\[
\begin{array}{ccc}
H^r((S^\infty \times Y)/T) & \xrightarrow{j^*} & H^r(M/T) \\
\downarrow{i_\infty^*} & & \downarrow{i_1^*} \\
H^r((S^\infty/T) \times F) & \xrightarrow{j_F^*} & H^r(P_{2n-1}(C) \times F),
\end{array}
\]

where \( i_1, i_\infty, j, j_F \) are natural inclusions. Since \( H^{\text{odd}}(Y) = 0 \), we see that (cf. [7])

1. \( i_\infty^* \) is injective, \( j^* \) is surjective and \( i_1^* \) is surjective for \( r > \dim Y \).

On the other hand, \( j_F^* \) is an isomorphism for \( r \leq 4n-2 \), and hence

2. \( i_1^* \) is injective for \( r \leq 4n-2 \).

Also we prepare the following for later use. The proof is omitted.

**Lemma 2.2.** Let \( S \) be a closed connected smooth \( Sp(1) \) manifold. Let \( F \) be the fixed point set of the restricted \( T \) action on \( S \), where \( T \) is a closed toral subgroup of \( Sp(1) \). Suppose that \( \text{codim } F = 2 \) and \( F \) is not connected. Then there is an equivariant diffeomorphism : \( S = Sp(1)/T \times F_1 \), where \( F_1 \) is a connected component of \( F \).
2-A. Now we consider the case $X \sim P_a(H) \times P_b(C)$.

Proposition 2.3. Suppose $X \sim P_a(H) \times P_b(C)$, $1 \leq a \leq n-1$, $1 \leq b \leq 2n-1$, $2n \leq 2a+b \leq 4n-4$. Then either $a = n-1$ and $F^{(1)} \sim P_b(C)$, or $b = 2n-1$ and $F^{(1)} \sim S^2 \times P_a(H)$.

(Proof) The cohomology ring is as follows.

$$H^*(X) = \mathbb{Q}[u, v]/(u^{a+1}, v^{b+1}); \text{deg } u = 4, \text{deg } v = 2.$$  

We can express $e(p) = \alpha u + \beta v^2$; $\alpha, \beta \in \mathbb{Q}$, where $p : S^{4n-1} \times F^{(1)} \to X$ is the principal $Sp(1)$ bundle. By definition, the $Sp(1)$ bundle $p$ is a pull-back of the canonical principal $Sp(1)$ bundle over $P_{n-1}(H)$, and hence $e(p)^n = 0$. Thus we obtain $\alpha \beta = 0$, by considering the term $u^{a} v^{2n-2a}$ in the expression of $e(p)^n$. On the other hand, we can prove $e(p) \neq 0$ by making use of the exact sequence $(A_i)$. Moreover we see, from $(A_i)$, that if $\beta = 0$ then $a = n-1$ and $F^{(1)} \sim P_b(C)$; if $\alpha = 0$ then $b = 2n-1$ and $F^{(1)} \sim S^2 \times P_a(H)$. q.e.d.

Now we consider the $Sp(1)$ action on $F^{(1)}$. Let $T$ be a toral subgroup of $Sp(1)$. Denote by $F$ the fixed point set of the restricted $T$ action on $F^{(1)}$. Since $\chi(F^{(1)}) \neq 0$, we see that $F$ is non-empty. We shall show the following.

Proposition 2.4. If $a = n-1$ and $F^{(1)} \sim P_b(C)$, then the $Sp(1)$ action on $F^{(1)}$ is trivial. If $b = 2n-1$ and $F^{(1)} \sim S^2 \times P_a(H)$, then $F \sim S^0 \times P_a(H)$ or $F \sim S^0 \times P_a(C)$. Moreover the induced homomorphism $i^* : H^2(F^{(1)}) \to H^2(F)$ is trivial.
(Proof) Put $Y = F_1$ in the diagram $(D-1)$. Let $t \in H^2(P_{2n-1}(C))$ and $w \in H^4(P_{n-1}(H))$ be the canonical generators. Then $\pi_2^*(w) = e(p)$ by definition. We see that $e(p) = \alpha u$, $\alpha \neq 0$ or $e(p) \neq \beta v^2$, $\beta \neq 0$ in Proposition 2.3.

Suppose first $e(p) = \alpha u$. Then $a = n-1$ and $F_1 \sim P_b(C)$. We can prove $M/T \sim P_{2n-1}(C) \times P_b(C)$, $b \leq 2n-2$ by making use of the Leray-Hirsch theorem, and hence the $T$ action on $F_1 \sim P_b(C)$ is trivial (cf. [E], Proposition 3.3). Therefore the $Sp(1)$ action on $F_1$ is trivial.

Suppose next $e(p) = \beta v^2$. Then $b = 2n-1$ and $F_1 \sim S^2 \times P_a(H)$. Put $u_1 = p_1^*(u)$, $v_1 = p_1^*(v)$ and $t_1 = \pi_1^*(t)$. We can apply the Leray-Hirsch theorem to the bundles $\pi_1_n_2$ in the diagram $(D-1)$, and we obtain

$$H^*(M/T) = \mathbb{Q} [t_1, u_1, v_1]/(u_1^{a+1}, v_1^{2n}, t_1^2 - \beta v_1^2), \beta \neq 0.$$  

Consider the diagram $(D-2)$ for $Y = F_1$. Let $u_2, v_2$ be homogeneous elements of $H^*(S^\infty \times F_1 / T)$ such that $j^*(u_2) = u_1$ and $j^*(v_2) = v_1$. Let $t$ be the canonical generator of $H^2(S^\infty / T) = H^2(P_{2n-1}(C))$. Then we can express $i^*_\infty(u_2) = t^2xf_0 + txf_1 + 1xf_2$, $i^*_\infty(v_2) = txg_0 + 1xg_1$, where $f_k, g_k$ are elements of $H^{2k}(F)$. Since $j^*_F i^*_\infty(\beta v_2^2) = i^*_F(\beta v_1^2) = i^*_F(t_1^2) = j^*_F(t^2x1)$, we obtain $g_0^2 = \beta^{-1}$ and $g_1 = 0$. Moreover we see that $g_0$ is not constant, and hence $F$ is not connected. Since $j^*_F i^*_\infty(u_2^{a+1}) = 0$ and $a+1 \leq n-1$, we obtain $f_0 = 0$ and hence $i^*_\infty(u_2) = txf_1 + 1xf_2$. Let $F_1$ (resp. $F_2$) be the union of connected components $F_\sigma$ of $F$ on which $g_0|_{F_\sigma}$ is positive (resp. negative). Then each element of $H^k((S^\infty \times F_S) / T)$ for $k > 4a+2$ is expressed as a polynomial of $tx1$ and $tx(f_1|_{F_S}) + 1x(f_2|_{F_S})$.
with rational coefficients for \( s = 1, 2 \), because \( H^*\left((S^\infty \times F(1)) / T \right) \) is generated by two elements \( u_2, v_2 \) as graded \( H^*(S^\infty / T) \)-algebra and \( i^* \) is surjective for \( k > 4a + 2 \). In particular, if \( f_1 | F_s \neq 0 \), then we can express

\[ t^{4a-1} x (f_1 | F_s^1) = \sum_j (c_j (t x (f_1 | F_s^1) + 1 x (f_2 | F_s^2))^j (t x 1)^{4a-2j}) \]

for \( c_j \in \mathbb{Q} \). Then we obtain \( c_0 = 0 \), \( c_1 = 1 \) and \( f_2 | F_s^2 = - c_2 (f_1 | F_s^1)^2 \). Therefore

\[ H^*(F_s) = \mathbb{Q}[x]/(x^{a+1}) ; \text{deg} x_s = 2 \text{ or } 4, \]

because \( f_{k}^{a+1} = 0 \) (\( k = 1, 2 \)) and \( \mathcal{K}(F_1) + \mathcal{K}(F_2) = \mathcal{K}(F_{(1)}) = 2a \).

If \( F_s \sim P_a(H) \) for some \( s \), then \( F \sim S^0 \times P_a(H) \) by Lemma 2.2.

Thus we obtain \( F \sim S^0 \times P_a(H) \) or \( F \sim S^0 \times P_a(C) \). Finally we shall show that \( i^* : H^2(F_{(1)}) \to H^2(F) \) is trivial for the case \( F \sim S^0 \times P_a(C) \). Consider the following commutative diagram:

\[
\begin{array}{ccc}
H^2(M/T) & \xrightarrow{k_1^*} & H^2(F_{(1)}) \\
\downarrow{i_1^*} & & \downarrow{i^*} \\
H^2(P_{2n-1}(C) \times F) & \xrightarrow{k_0^*} & H^2(F),
\end{array}
\]

We see that \( k_1^*(v_1) \) generates \( H^2(F_{(1)}) \) and \( i_1^*(v_1) = t x g_0 \), and hence \( i^* k_1^*(v_1) = k_0^*(t x g_0) = 0 \). Thus \( i^* : H^2(F_{(1)}) \to H^2(F) \) is trivial. q.e.d.

Suppose \( F \sim S^0 \times P_a(H) \). Then by Lemma 2.2, there is an equivariant diffeomorphism \( F_{(1)} = Sp(1)/T \times Y_2 \), where \( Y_2 \) is a connected component of \( F \). Thus we obtain an equivariant diffeomorphism:

\[ X = X_{(1)} = (S^{4n-1} \times F_{(1)}) / Sp(1) = P_{2n-1}(C) \times Y_2. \]

Consequently we obtain the following.
Theorem 2.5. Let $X$ be a closed orientable manifold with a non-trivial smooth $\text{Sp}(n)$ action. Suppose $n \geq 7$, $X = X(0) \cup X_{(1)}$ and $X \sim P_a(H) \times P_b(C)$, $1 \leq a \leq n-1$, $1 \leq b \leq 2n-1$, $2n \leq 2a+b \leq 4n-4$. Then there are three cases:

(a) $a = n-1$ and $X$ is equivariantly diffeomorphic to $P_{n-1}(H) \times Y_1$, where $Y_1$ is a closed orientable manifold such that $Y_1 \sim P_b(C)$,

(b) $b = 2n-1$ and $X$ is equivariantly diffeomorphic to $P_{2n-1}(C) \times Y_2$, where $Y_2$ is a closed orientable manifold such that $Y_2 \sim P_a(H)$,

(c) $b = 2n-1$ and $X$ is equivariantly diffeomorphic to $(S^{4n-1} \times Y_3)/\text{Sp}(1)$, where $Y_3$ is a closed orientable $\text{Sp}(1)$ manifold such that $Y_3 \sim S^2 \times P_a(H)$, $F \sim S^0 \times P_a(C)$ and $i^* : H^2(Y_3) \to H^2(F)$ is trivial, where $F$ denotes the fixed point set of the restricted $T$ action on $Y_3$. Conversely, if $Y_3$ satisfies the above conditions, then $(S^{4n-1} \times Y_3)/\text{Sp}(1) \sim P_{2n-1}(C) \times P_a(H)$ for $a \leq n-2$.

Remark. In the above theorem 2.5, it remains to prove the final statement in the case (c). But the proof is omitted here. To prove this, the condition that $i^* : H^2(Y_3) \to H^2(F)$ is trivial is not necessary for $a > 1$, but it can not be omitted for $a = 1$.

2-B. Next we consider the case $X \sim P_a(C) \times P_b(C)$. By the same way as in the case 2-A, we have the following Propositions. The proofs are omitted.

Proposition 2.6. Suppose $X \sim P_a(C) \times P_b(C)$, $1 \leq b \leq a < 2n \leq a+b \leq 4n-3$. Then $a = 2n-1$ and $F(1) \sim S^2 \times P_b(C)$. 

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Proposition 2.7. \( F \sim S^0 \times P_b(C) \).

Consequently we obtain the following.

Theorem 2.8. Let \( X \) be a closed orientable manifold with a non-trivial smooth \( \text{Sp}(n) \) action. Suppose \( n \geq 7 \), \( X = X_{(0)} \cup X_{(1)} \) and \( X \sim P_a(C) \times P_b(C) \), \( 1 \leq b \leq a \leq 2n \leq a+b \leq 4n-3 \). Then \( a = 2n-1 \) and \( X \) is equivariantly diffeomorphic to \( P_{2n-1}(C) \times Y_0 \), where \( Y_0 \) is a closed orientable manifold such that \( Y_0 \sim P_b(C) \).

3. Cohomologies of certain homogeneous spaces

In this section, we give the cohomologies of \( V_{n,2}/G = \text{Sp}(n)/\text{Sp}(n-2) \times G \) for certain closed connected subgroups \( G \) of \( \text{Sp}(n) \). The results are as follows. The actual proofs are omitted here (see [8]).

Lemma 3.1. \( H^*(V_{n,2}/\text{Sp}(1) \times \text{Sp}(1)) = Q[u,v]/(u^n, \zeta u^{i}v^{n-1-i}) \), \( \deg u = \deg v = 4 \).

Lemma 3.2. \( H^*(V_{n,2}/T^2) = Q[x,y]/(x^{2n}, \zeta x^{2i}y^{2n-2-2i}) \), \( \deg x = \deg y = 2 \).

Lemma 3.3. The graded algebra \( H^*(V_{n,2}/\text{Sp}(2)) \) is isomorphic to the subalgebra of \( Q[u,v]/(u^n, \zeta u^{i}v^{n-1-i}) \), consisting of symmetric polynomials, where \( \deg u = \deg v = 4 \).

Lemma 3.4. The graded algebra \( H^*(V_{n,2}/U(2)) \) is isomorphic to the subalgebra of \( Q[x,y]/(x^{2n}, \zeta x^{2i}y^{2n-2-2i}) \), consisting of symmetric polynomials, where \( \deg x = \deg y = 2 \).

Lemma 3.5. The graded algebra \( H^*(V_{n,2}/U(1) \times \text{Sp}(1)) \) is isomorphic to the subalgebra of \( Q[x,y]/(x^{2n}, \zeta x^{2i}y^{2n-2-2i}) \) generated by \( x^2, y \).
From these lemmas, we have

Proposition 3.6. Let $G$ be one of $T^2$, $U(2)$ and $U(1) \times \text{Sp}(1)$. Let $w_1, w_2$ be any non-zero homogeneous elements of $H^*(V_n, 2G)$ such that $\deg w_k = 2k$. Then $w_1^{2n-1}$ and $w_2^{n-1}$ are non-zero elements.

4. Finish of the proof

Throughout this section, suppose that $n \geq 7$ and $X$ is a closed orientable manifold with a non-trivial smooth $\text{Sp}(n)$ action, and $X \simeq P_a(C) \times P_b(C)$ for some $a, b$ such that

$1 \leq b \leq a < 2n \leq a+b \leq 4n-3$,

or $X \simeq P_c(H) \times P_d(C)$ for some $c, d$ such that

$1 \leq c \leq n-1, 1 \leq d \leq 2n-1$ and $2n \leq 2c+d \leq 4n-4$.

Then, from the results in the section 3, we can show that $X(2)$ and $X(3)$ are empty sets. That is, we can prove the following Propositions. We shall give here the outline of the proofs of those Propositions (see [8] for the details).

Proposition 4.1. $X \not\simeq X_{(k)}$; $k = 2, 3$.

(Outline of the proof) Suppose $X = X_{(k)}$. Then $X = (\text{Sp}(n)/\text{Sp}(n-k) \times F_{(k)}))/\text{Sp}(k)$. In particular, we obtain $\mathcal{K}(X) = nc_k\mathcal{K}(F_{(k)})$. From this fact, we see that $k \neq 3$ and the possibilities remain only in the following cases:

(a) $\dim F(2) = 8$, $\mathcal{K}(F(2)) = 8$; $(a,b) = (2n-1,2n-3)$,
(b) $\dim F(2) = 6$, $\mathcal{K}(F(2)) = 4$; $(c,d) = (n-1,2n-3)$, $(n-2,2n-1)$,
(c) $\dim F(2) \leq 4$. 

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If \( \dim F(2) \leq 4 \), then \( X = V_{n,2}/\text{Sp}(1) \times \text{Sp}(1) \) or \( V_{n,2}/\text{Sp}(2) \times F(2) \), and hence (c) does not happen by Lemmas 3.1 and 3.3.

In the cases (a), (b), if the \( \text{Sp}(2) \) action on \( F(2) \) is transitive, then \( X = V_{n,2}/T^2 \), \( V_{n,2}/\text{U}(2) \) or \( V_{n,2}/\text{U}(1) \times \text{Sp}(1) \), and hence such cases do not happen by Proposition 3.6. So the remainder of the possibilities is as follows: in each case below, \( \text{Sp}(2) \) action on \( F(2) \) is not transitive,

(a)' the case (a) and the restricted \( G \) action on \( F(2) \) has a fixed point, where \( G = \text{U}(2) \) or \( \text{U}(1) \times \text{Sp}(1) \),

(b)' the case (b) and the \( \text{Sp}(2) \) action on \( F(2) \) is trivial,

(c)' the case (b) and the \( \text{Sp}(2) \) action on \( F(2) \) has no fixed point,

(d)' the case (b) and the \( \text{Sp}(2) \) action on \( F(2) \) has a fixed point but this action is not trivial.

Consider the case (a)'. Then the natural projection \( \overline{\pi}_1: (V_{n,2} \times F(2))/G \rightarrow V_{n,2}/G \) has a cross section \( s \), and we have the following commutative diagram:

\[
\begin{array}{ccc}
(V_{n,2} \times F(2))/G & \xrightarrow{s} & V_{n,2}/G \\
\downarrow q & & \downarrow p \\
X = (V_{n,2} \times F(2))/\text{Sp}(2) & \xrightarrow{\overline{\pi}_2} & V_{n,2}/\text{Sp}(2),
\end{array}
\]

\( \overline{\pi}_1, \overline{\pi}_2, p, q \) : natural projections.

From this diagram and the cohomologies of \( V_{n,2}/\text{Sp}(2) \) and \( V_{n,2}/G \) (see Lemmas 3.1, 3.4 and 3.5), we can see that if \( X \sim \text{P}_a(C) \times \text{P}_b(C) \), then \( p^*|H^4(V_{n,2}/\text{Sp}(2)) \) is not injective.
This is a contradiction. If \( F_{(2)} \) is such as in the case (b)', then \( X = V_{n,2}/Sp(2) \times F_{(2)} \), and hence such a case does not happen by Lemma 3.3.

If \( F_{(2)} \) is such as in the case (c)', then we can see that the identity component of an isotropy subgroup is conjugate to \( Sp(1) \times Sp(1) \) and that the fixed point set \( F \) of the restricted \( Sp(1) \times Sp(1) \) action on \( F_{(2)} \) is a closed orientable surface with \( \chi(F) = 4 \) and \( F \) has at most two components. Therefore \( X = (V_{n,2}/Sp(1) \times Sp(1)) \times S^2 \), and hence such a case does not happen by Lemma 3.1.

Finally consider the case (d)'. Then we see that the fixed point set \( F' \) of the \( Sp(2) \) action is 1-dimensional, and the identity component of the other isotropy subgroup is conjugate to \( Sp(1) \times Sp(1) \). Let \( U \) be a closed tubular neighbourhood of \( F' \), and let \( F'' \) be the fixed point set of the restricted \( Sp(1) \times Sp(1) \) action on \( F_{(2)} - \operatorname{int} U \). Then we see that \( F'' \) is a compact orientable surface with \( \chi(F'') = 4 \), \( F'' \) has at most two components and each component of \( F'' \) has a non-empty boundary. Such a case does not happen, because \( \chi \leq 1 \) for each compact connected orientable surface with non-empty boundary.

**Proposition 4.2.** If \( X_{(1)} \) is non-empty, then \( X_{(2)} \) is empty.

*(Outline of the proof)* Suppose that both of \( X_{(1)}, X_{(2)} \) are non-empty. Then \( X = X_{(1)} \cup X_{(2)} \) and \( \operatorname{codim} F_{(1)} = 8n-8 \).
by Propositions 1.3, 1.4. Since \( \dim X \leq 8n-6 \), we obtain \( \dim F(1) = 0 \) or 2. Then we have the following possibilities:

(a) the \( \text{Sp}(1) \) action on \( F(1) \) is non-trivial,
(b) the \( \text{Sp}(1) \) action on \( F(1) \) is trivial, and

\[(b.1) \dim F(1) = 0 ; (a,b) = (2n-1,2n-3) \text{ or } (2n-2,2n-2),
\[c,d) = (n-1,2n-2),
\]
or \[(b.2) \dim F(1) = 2 ; (a,b) = (2n-1,2n-2).
\]

For each case above, we first investigate the possibilities of the orbit types. And, from the results (in the section 3) about the cohomologies of such orbits, we deduce that those cases do not happen. For example, consider the case (a). Then \( \dim F(1) = 2 \), and \( X \sim P_{2n-1}(C) \times P_{2n-2}(C) \). Considering the slice representation at a point of \( F(1) \), we see that the \( \text{Sp}(n) \) action on \( X \) has a codimension one orbit, and hence \( X \) is a union of closed invariant tubular neighbourhoods of just two non-principal orbits (cf. [4]). Calculating the Euler characteristics, we see that two non-principal orbits are \( P_{2n-1}(C) \) and \( V_n,2/T^2 \). Since \( \text{codim} \ P_{2n-1}(C) = 4n-4 \) in \( X \), the inclusion \( i : V_n,2/T^2 \to X \) induces an isomorphism \( i^* : H^2(X) \to H^2(V_n,2/T^2) \), and hence \( x^{2n-1} \neq 0 \) for each non-zero element \( x \in H^2(X) \) by Proposition 3.6. This is a contradiction.

Similarly, we can deduce a contradiction for the case (b.1). The proof for the case (b.2) is a little more complicated than that for the above two cases, but we omit it here.
Under the consideration of the section 1, we obtain the main theorem stated in Introduction, by combining Theorems 2.5, 2.8 and Propositions 4.1, 4.2. The full proofs of the results in this paper will appear in [8].

References


