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<th>Rational Smith Equivalence of Representations (TRANSFORMATION GROUPS AND REPRESENTATION THEORY)</th>
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<td>Author(s)</td>
<td>Petrie, Ted</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1983), 501: 74-85</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1983-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/103678">http://hdl.handle.net/2433/103678</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Rational Smith Equivalence of Representations

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§ 1. Statement of results.

A famous theorem of Atiyah-Bott and Milnor asserts that if a finite group $G$ acts smoothly on a closed rational homotopy sphere $\Sigma$ with $\Sigma^G = p \cup q$, then the representations of $G$ on $T_p\Sigma$ and $T_q\Sigma$ are equal provided the action is semi-free. This is a report on joint work in progress with K.H. Dovermann where we show that for many cyclic groups of odd order, the result is false if the semi-free assumption is deleted. This is a prelude to our study where rational homotopy sphere is replaced by homotopy sphere. The author wishes to emphasize that proofs of results stated here exist in outline form only; so there may be some changes before the results obtain final form.

Let $V$ be a representation of $G$ and $E$ an acyclic $G$ space on which $G$ acts freely. A smooth $G$ manifold $W$ is said to be $V$ framed if there is a stable $G$ vector bundle isomorphism $\beta : E \times TW \to E \times W \times V$. These bundles are $G$ vector bundles over $E \times W$. There is an obvious notion of framed cobordism for $V$ framed manifolds. Such a cobordism is said to be $\text{rel}(W^H|H \subseteq G \ H \neq 1)$ if it is a product on $H$ fixed sets for $H \neq 1$. By definition $W$ is framed if it is framed for some $V$. 
Let $U$ and $V$ be representations of $G$. Write $U \varnothing V$ if there is a rational homotopy sphere $\Sigma$ with $G$ action such that $\Sigma^G = p \cup q$, $T_p \Sigma = U$, $T_q \Sigma = V$. We define a set $S_1$ of divisors of $|G|$, a subgroup $\overline{R}(G)$ of the complex representation ring of $G$ and a homomorphism

$$\lambda : \overline{R}(G) \to \bigotimes_{d \in S_1} \mathbb{C}^X/\mathbb{Z}_2 = \Gamma.$$ 

Here $\mathbb{C}^X = \mathbb{C} - \{0\}$ and $\mathbb{Z}_2$ is the subgroup of $\mathbb{M}^X$ generated by $(-1, -1, \ldots, -1) = -1$. Note $\Gamma$ is a multiplicative group.

**Theorem A:** If $z \in \text{Ker}(\lambda)$, then there are representations $U$ and $V$ of $G$ such that $r(z) = U - V$ and $U \varnothing V$. Here $r : R(G) \to R^n(G)$ denotes "realification".

For cyclic groups with at least four distinct primes dividing $|G|$, $\text{Ker} \lambda$ is non zero. In fact it's usually large. The main geometric ingredient in the proof of Theorem A is this theorem:

**Theorem B:** Let $G$ be cyclic of odd order. Suppose $W$ is a closed $4k$ dimensional framed manifold with $G$ action such that

i) $\dim W^G = 0$

ii) For $H \subset G$, $H \neq 1$, the Euler characteristic of $W^H \chi(W^H)$ is $2$ and $\dim W^H < \frac{1}{2} \dim W$

iii) $\text{Sign } (G, W) = 0$

Then $W$ is framed cobordant to $W'$ rel$\{W^H | H \subset G, H \neq 1\}$ and $W'$ is a rational homotopy sphere.
Corollary C: \( W^G \) consists of 2 points \( p \) and \( q \) and 
\[ T_p^G \sim T_q^G. \]

\section{Outline of ideas used in theorems A and B.}

We briefly indicate the ideas used in A) and B). This requires additional notation. Let \( \Lambda \) be \( \mathbb{Z} \) or \( \mathbb{Q} \) and \( n \) be an even integer. Let \( W_n(G, \Lambda) \) be the equivariant Witt ring denoted by \( W_n(\Lambda, G) \) in [ACH]. Briefly \( W_n(G, \Lambda) \) consists of equivalence classes of pairs \( (M, \phi) \) where \( M \) is a \( \Lambda \) torsion free \( \Lambda(G) \) module and \( \phi \) is a non singular, \( G \) invariant \( \Lambda \) valued bilinear form which satisfies 
\[ \phi(x, y) = (-1)^{n/2}\phi(y, x) \] for \( x, y \in M \). If \( W \) is a closed manifold of dimension \( n \) with \( G \) acting preserving orientation, then \( [W]_\Lambda \in W_n(G, \Lambda) \) is the class of 
\( (H^{n/2}(W, \Lambda)/\text{Torsion}, \phi_W) \) where \( \phi_W \) is the cup product bilinear form on \( W \). We remark that \( [W]_\Lambda \) depends only on the \( G \) cobordism class of \( W \). Note this key observation:

2.1 \( [W]_\mathbb{Z} = 0 \) if \( W \) is a rational homology sphere. In the case \( |G| \) is odd [ACH] give necessary and sufficient conditions that \( [W]_\mathbb{Z} = 0 \) which we exploit. To do this we henceforth suppose \( G \) is an odd order cyclic group and \( W \) is a closed oriented smooth \( G \) manifold of dimension \( 4k \) and in addition we assume \( \dim W^G = 0 \). In this case there is a simple formula for the torsion signatures \( \{w_p(G, W) \mid p \text{ is a prime which divides } |G|\} \). Note that in the notation of [ACH] \( w_p(G, W) = f(T, p) \) where \( T \) generates \( G \). See [ACH] pages 149-151. Let \( p \) be a prime which divides \( |G| \) and let \( P \) be
the $p$ Sylow subgroup of $G$. Call $p$ good if there is no integer $x$ such that $-1 \equiv p^x \mod |G/H|$; otherwise $p$ is bad.

**Lemma 2.3.** Under the above assumptions on $W$, $w_p(G, W) = 0$
if $p$ is good and $w_p(G, W) = \sum_{x \in W^G} \frac{1}{2}(\dim T_x W - \dim T_x W^P) \mod 2$
if $p$ is bad. (See 2.20)

**Proof:** This is immediate from [ACH, 1.8 p.141 and 3.5 p.149].

We emphasize that $w_p(G, W) \in \mathbb{Z}_2$ for each prime $p$ which divides $|G|$. These invariants are all functions of $[W]_2$.

**Theorem 2.4.** [ACH, 3.6 p.151] $[W]_2 = 0$ iff $\text{Sign}(G, W) = 0$
and $w_p(G, W) = 0$ for all $p$ which divide $|G|$.

**Corollary 2.5.** If $W$ is a rational homology sphere with $W^G = x \cup y$ (2 points), then $\frac{1}{2}(\dim T_x W^P - \dim T_y W^P) = 0(2)$ for each $p$ Sylow subgroup for which $p$ is bad.

**Proof:** This is immediate from 2.3 using the fact that $\dim T_x W$ and $\dim T_x W^P$ are even.

Corollary 2.5 gives an especially simple necessary condition that the representations $U$ and $V$ of $G$ occur as
$(T_x W, T_y W)$ for some smooth action of $G$ on a rational homology sphere $W$ with $W^G = x \cup y$. Actually much more stringent necessary conditions come from the condition $\text{Sign}(G, W) = 0$.

In fact if we add the condition that $W$ be framed, all $w_p(G, W)$ vanish. Here is the argument:

**Theorem 2.6.** Suppose $W$ is framed and $W^G = x \cup y$, then $w_p(G, W) = 0$ for all $p$ which divide $|G|$.
Proof: The results of Atiyah in [A] assert:

\[ + \) Ker(R(G) \to K_G(E) = K(E/G) = \hat{R}(G) \]
\[ = Ker(R(G) \xrightarrow{\text{res}} \underset{P \text{ Sylow}}{\coprod} R(P)) \]

(The \( P \) component of \( \text{res} \) is \( (\text{res})_P = \text{res}_P \) where
\( \text{res}_P : R(G) \to R(P) \) is restriction to \( P \subset G \).) Clearly
\( T_x^W - T_y^W \in Ker(R(G) \xrightarrow{\text{res}} \underset{P \text{ Sylow}}{\coprod} R(P)) \) if \( W \) is framed; so
\( T_x^W - T_y^W \in Ker(R(G) \xrightarrow{\text{res}} \underset{P \text{ Sylow}}{\coprod} R(P)). \) Now the assertion
\( w_p(G, W) = 0 \) follows from 2.3. (See 2.19)

**Corollary 2.8.** Let \( W \) be a framed \( G \) manifold with
\( W^G = p \cup g. \) Then \([W]_2 = 0 \) iff \( \text{Sign}(G, W) = 0. \)

Now we discuss framed manifolds and equivariant surgery. The process of equivariant framed surgery is well understood when \( G \) acts freely on \( W \). (See e.g. [W]). We treat this case first. Suppose \( G \) acts freely on \( W \) and
\( \beta : TW \cong (W \times V) \) is a stable \( G \) vector bundle isomorphism for some representation \( V \) of \( G \). Call \( \beta \) a strong framing of \( W \). Then for any \( x \in \pi_j(W) \) \( j \leq n/2 \) \( (n = \text{dim } W) \), there is a \( G \) immersion (imbedding if \( j < n/2) \)
\( \iota : G \times S^j \times D^{n-j} \to W \) such that \( \iota|S^j \) represents \( x \). If \( \iota \) is a \( G \) imbedding, there is a strong framing \( \beta' \) of \( W' = W\text{-interior } (G \times S^j \times D^{n-j}) \cup G \times D^{j+1} \times S^{n-j-1} \) which agrees with \( \beta \) over \( W\text{-interior } (G \times S^j \times D^{n-j}). \) This construction \( (W, \beta) \to (W', \beta') \) is called equivariant surgery and may be used to kill \( \pi_j(W) \) for \( j < n/2 \). In fact \( W \) is
strongly framed cobordant to a manifold \( W' \) with \( \pi_j(W') = 0 \) for \( j < n/2 \). For elaboration of these ideas, see [PR].

This discussion generalizes as follows:

**Lemma 2.9.** Suppose \( W \) is framed and \( \dim W^H < \frac{1}{2} \dim W \) whenever \( H \neq 1 \). Then \( W \) is framed cobordant rel\{\( W^H | H \neq 1 \)\} to a manifold \( W' \) with \( \pi_j(W') = 0 \) for \( j < n/2 \). (\( n = \dim W \)).

**Proof:** Here is an outline: Let \( W^* = W - \bigcup_{H \neq 1} W^H \); so \( G \) acts freely on \( W^* \). This means the projection of \( E \times W^* \) on \( W^* \) is a \( G \) homotopy equivalence and this means that framing and strong framing of \( W^* \) is the same notion. Next note that the inclusion \( W^* \to W \) induces an isomorphism in homotopy in dimensions not exceeding \( n/2 \); so any class \( x \in \pi_j(W) \) \( j < n/2 \) comes from a class \( x' \in \pi_j(W^*) \). Now note that the framing of \( W \) gives a framing of \( W^* \); so \( W^* \) is strongly framed. Thus we may apply the above discussion to \( W^* \) and \( x' \).

This provides a \( G \) imbedding of \( G \times S^j \times D^{n-j} \) in \( W^* \subset W \), so we can form \( W' = W \)-interior \((G \times S^j \times D^{n-j}) \cup G \times D^{j+1} \times S^{n-j-1}\) as before. (Observe that \( W^H = W^H \) for all \( H \neq 1 \). This is the reason that the cobordism asserted is rel\{\( W^H | H \neq 1 \)\}.

**Lemma 2.10.** Suppose \( \chi(W^H) = 2 \) for all \( H \neq 1 \) and \( \tilde{H}_j(W, \mathcal{Q}) = 0 \) for \( j < n/2 \) \( n = \dim W \). Then \( H_{n/2}(W, \mathcal{Q}) \) and \( H^{n/2}(W, \mathcal{Q}) \) are free \( \mathcal{Q}(G) \) modules.

**Proof:** By hypothesis \( \dim W^G = 0 \); so \( W^G \) is non empty. Let \( x \in W^G \) and let \( V \) be the representation \( T_x W \). Set \( n = 2k \) and \( S = S(V \oplus \mathbb{R}) \) where \( \mathbb{R} \) is the trivial one dimensional real
representation and $S(V \oplus \mathbb{R})$ is the unit sphere of $V \oplus \mathbb{R}$.

The Thom map $f : W \rightarrow S$ obtained by collapsing the exterior of an invariant disk centered at $x$ has degree 1. Let $M_f$ be the mapping cone of $f$. Then $\chi(M_f^H) = 1$ for $H \neq 1$ (because degree $f = 1$ and $\chi(W^H) = \chi(S^H) = 2$ for $H \neq 1$). In addition $\tilde{H}_i(M_f, \mathbb{Q}) = 0$ for $i \neq k + 1$. These two properties imply that $H_{k+1}(M_f, \mathbb{Q}) \cong H_k(W, \mathbb{Q})$ is a free $\mathbb{Q}(G)$ module. (See [0])

The obstruction to converting a framed manifold $W$ satisfying:

2.11 $\dim W^H < \frac{1}{2} \dim W$ and $\chi(W^H) = 2$ for $H \neq 1$.

into a rational homology sphere $\sum$ using equivariant surgery is an element $\sigma(W) \in L(\mathbb{Q}(G))$. Here $L(\mathbb{Q}(G))$ is an abbreviation for the Wall group $L^h_n(\mathbb{Q}(G), 1)$. Briefly this is an abelian group consisting of equivalence classes of triples $(M, \lambda, \mu)$ where $M$ is a free $\mathbb{Q}(G)$ module, $\lambda$ is a non singular, $G$ invariant, $\mathbb{Q}$ valued bilinear form which satisfies $\lambda(x, y) = (-1)^{n/2} \lambda(y, x)$ for $x, y \in M$ and $\mu$ is an associated quadratic form. (See [W, §5] for notation). If $W$ satisfies 2.11, it is framed cobordant to a manifold $W''$ which also satisfies 2.11 and in addition, $\pi_j(W'') = 0$ for $j < n/2$ (2.9). By 2.10 $M = H^{n/2}(W'', \mathbb{Q})$ is a free $\mathbb{Q}(G)$ module. Then $\sigma(W)$ is the class of $(H^{n/2}(W'', \mathbb{Q}), \phi_w, \mu_w)$ where $\mu_w$ is the self intersection form of $W''$ (See [W, §5]).

There is an obvious homomorphism $\rho : L(\mathbb{Q}(G)) \rightarrow W_n(\mathbb{Q}, \mathbb{Q})$ which sends $\sigma(W)$ to $[W]_\mathbb{Q}$. Because $n$ is 0 mod 4, $\rho$ is
injective. We can now give a proof of Theorem B.

Proof of Theorem B: By i) and ii), \( W^G \) consists of two points \( x \) and \( y \). By 2.8 \( [W]^Z = 0 \) and this implies that \( [W]^Q = 0 \). But \( [W]^Q = \rho \sigma(W) \). Since \( \rho \) is injective, \( \sigma(W) = 0 \). Since \( \sigma(W) \) is the obstruction to converting \( W \) to a rational homology sphere \( \sum \) and since \( \sigma(W) = 0 \), \( \sum \) exists.

Now we turn to the discussion of Theorem A. We view the cyclic group \( G \) as the subgroup of \( \mathbb{C}^X = \mathbb{C} - 0 \) consisting of the \( |G| \)th roots of unity. Let \( t^i \) denote the complex one dimensional representation of \( G \) on which \( g \in G \) acts on \( v \in t^i \) by \( g(v) = g^i \cdot v \) i.e. complex multiplication by \( g^i \).

A complex representation \( V \) of \( G \) may be uniquely written as

\[
V = \sum_{i=0}^{\lfloor \frac{|G|-1}{|G|} \rfloor} a_i t^i
\]

for some integers \( a_i \geq 0 \). For \( g \in G \),

\[
V^g = \{ v \in V | gv = v \}.
\]

When \( V^g = 0 \), we can define this complex number:

\[
2.12 \quad v(V)(g) = \prod_{i=0}^{\lfloor \frac{|G|-1}{|G|} \rfloor} \left( \frac{1+g^i}{1-g^i} \right) a_i \in \mathbb{C}^X.
\]

The assumption \( V^g = 0 \) means the denominator does not vanish. These complex numbers appear in the Atiyah Singer index formula for \( \text{Sign}(g, W) \) when \( \dim W^g = 0 \). Here is a discussion of this point. Suppose \( W^g = W^G \). (By hypothesis \( \dim W^G = 0 \).)

Let \( x \in W^G \). Since \( G \) preserves orientation, there is complex representation of \( G \) whose realification is \( T_x W \). Choose one \( T_x W \) for which the orientation given by the complex structure agrees with the given orientation on \( T_x W \). Then
2.13 \[ \text{Sign}(g, W) = \sum_{x \in W} \nu(\tilde{T}_x W)(g). \]

We remark that if \( V \) and \( V' \) are two complex representations whose realifications are both \( T_x W \), then \( \nu(V')(g) = \pm \nu(V)(g) \); so there is a sign ambiguity for the right hand side of 2.13 as a function of the real representation \( T_x W \). This is ambiguity disappears when orientation is accounted for in the way mentioned. Another relevant elementary point is that if \( r(V) = T_x W \), there is a complex representation \( V' \) such that \( r(V') = r(V) \) and \( \nu(V')(g) = -\nu(V)(g) \) for all \( g \) for which \( dV' = 0 \).

Theorem B is used in the proof of Theorem A. To use Theorem B for this purpose we need to produce a framed manifold \( W \) with \( W^G = x \cup y \) (two points) and \( \text{Sign}(G, W) = 0 \). Let \( V = \tilde{T}_x W \) and \( U = \tilde{T}_y W \) and let \( g \) be an element of \( G \) for which \( W^g = 0 \). Then \( \nu^g = 0 = V^g \) and

\[ \text{Sign}(g, W) = \nu(V)(g) + \nu(U)(g). \]

so

2.14 \[ 0 = \text{Sign}(g, W) \iff \nu(V)(g)/\nu(U)(g) = -1. \]

In summary we have obtained these conditions on two representations \( U \) and \( V \) of \( G \):

**Lemma 2.15.** Let \( W \) be a framed \( G \) manifold with \( W^G = x \cup y \), \( \tilde{T}_x W = V \), \( \tilde{T}_y W = U \) and \( \text{Sign}(G, W) = 0 \). Then

(i) \( V - U \in \text{Ker}(R(G) \xrightarrow{\text{res}} \prod P \text{Sylow}) \) and

(ii) \( \nu(V)(g)/\nu(U)(g) = -1 \) whenever \( W^g = W^G \).
(Note this implies \( U^g = V^g = 0 \).)

Lemma 2.15 and the discussion preceding it lead to sufficient conditions that two representations \( U \) and \( V \) occur (stably) as \( T_x^S \) and \( T_y^S \) for some rational homology sphere \( S \) with \( x^g = x \cup y \). We discuss this point.

Let \( S_1 \) be the set of divisors \( d \) of \( |G| \) such that \( |G|/d \) is a prime power and let \( S_2 \) be the set of divisors \( d \) of \( |G| \) such that \( |G|/d \) is divisible by at most three distinct primes. Let

\[
\mathcal{R}(G) = \{ U - V \in \mathcal{R}(G) \text{ such that i-iii hold}\}.
\]

i) \( V^g = U^g = 0 \) whenever \( g \in G \) and \( |g| \in S_1 \)

ii) \( \dim V^g = \dim U^g \) whenever \( |g| \in S_2 \)

iii) \( V - U \in \text{Ker}(R(G) \xrightarrow{\text{res}} \prod \text{R}(P)) \)

We define the homomorphism \( \lambda \) in Theorem A.

\[
\lambda : \mathcal{R}(G) \to \prod_{S_1} \mathbb{C}^x/\mathbb{Z}_2.
\]

If \( d \in S_1 \), the \( d \)th coordinate of \( \lambda \) is

\[
\lambda_d(V - U) = \nu(V)(g)/\nu(V)(g) \quad g = \exp(2\Pi i/d) \in G.
\]

We can only very briefly discuss the points of the proof of Theorem A. If \( z \in \text{Ker} \lambda \), there is a manifold \( W \) satisfying the assumptions of Theorem B and in addition \( W^g = x \cup y \) and \( T_x W - T_y W = r(z) \). Theorem A follows from this and Corollary C. Here are the essential points: There are complex representations \( U \) and \( V \) of \( G \) satisfying 2.16 i-iii and in
addition

2.17 \quad i) \quad r(V - U) = r(z) \\
ii) \quad \lambda(V - U) = -\frac{1}{2}.

(If \( \lambda(z) = -1 \), \( V - U = z \). If \( \lambda(z) = 1 \), then \( V - U \) is not \( z \), but \( r(V - U) = r(z) \). This is related to the discussion after 2.13.) Use 2.16 i) and the methods of [P] to produce manifolds \( X(V) \) and \( X(U) \) with these properties:

2.18 \quad i) \quad X(V)^G = x(\text{one point}), \ X(U) = y \ (\text{one point}) \\
ii) \quad X(V)^g = X(V)^G \text{ and } X(U)^g = X(U)^G \text{ whenever } |g| \in S_1. \\
iii) \quad TX(V) \text{ and } TX(U) \text{ are stably } G \text{ isomorphic to } X(V) \times V \text{ and } X(U) \times U \text{ respectively.}

By 2.16 iii) and 2.6 +), \( W = X(V) \sqcup X(U) \) is framed; moreover, \( x^g = x \cup y \) whenever \( |g| \in S_1 \) and \( T_xW = V, \ T_yW = U \) by construction. (Note 2.18 iii) which implies \( T_xW = r(V) \) and \( T_yW = r(U) \).) Whenever \( |g| \in S_1 \), \( \text{Sign}(g, W) = 0 \) because \( \lambda|g|(V - U) = v(V)(g)/v(U)(g) = -1 \). See 2.14. Of course the condition \( \text{Sign}(G, W) = 0 \) requires \( \text{Sign}(g, W) = 0 \) for all \( g \in G \) not just \( g \in G \) with \( |g| \in S_1 \). The fact that \( \text{Sign}(g, W) = 0 \) for \( |g| \notin S_1 \) and the other properties of \( W \) required for Theorem B are consequences of other properties of the construction of \( W \) which we omit.

2.19 \quad At some points in text we do not distinguish between real and complex representations. Since \( G \) is cyclic this should cause no problem.

2.20 \quad The assumption \( \dim W = 4k \) may be dropped.
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