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Rational Smith Equivalence of Representations

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§ 1. Statement of results.

A famous theorem of Atiyah-Bott and Milnor asserts that if a finite group G acts smoothly on a closed rational homotopy sphere Σ with $\Sigma^G = p \cup q$, then the representations of G on $T_p\Sigma$ and $T_q\Sigma$ are equal provided the action is semi-free. This is a report on joint work in progress with K.H. Dovermann where we show that for many cyclic groups of odd order, the result is false if the semi-free assumption is deleted. This is a prelude to our study where rational homotopy sphere is replaced by homotopy sphere. The author wishes to emphasize that proofs of results stated here exist in outline form only; so there may be some changes before the results obtain final form.

Let V be a representation of G and E an acyclic G space on which G acts freely. A smooth G manifold W is said to be V framed if there is a stable G vector bundle isomorphism $\beta : E \times TW \to E \times W \times V$. These bundles are G vector bundles over $E \times W$. There is an obvious notion of framed cobordism for V framed manifolds. Such a cobordism is said to be rel$\{W^H | H \subset G \quad H \neq 1\}$ if it is a product on H fixed sets for $H \neq 1$. By definition W is framed if it is framed for some V.

Let U and V be representations of G. Write $U \underset{\mathbb{Q}}{\sim} V$ if there is a rational homotopy sphere Σ with G action such that $\Sigma^G = p \cup q$ $T_p\Sigma = U$, $T_q\Sigma = V$. We define a set S_1 of divisors of $|G|$, a subgroup $\overline{R}(G)$ of the complex representation ring of G and a homomorphism

$$\lambda : \overline{R}(G) \to \prod_{d \in S_1} \mathbb{C}^x/\mathbb{Z}_2 = \Gamma.$$

Here $\mathbb{C}^x = \mathbb{C} - 0$ and $\mathbb{Z}_2$ is the subgroup of $\Pi\mathbb{C}^x$ generated by $(-1, -1, \cdots -1) = -\underline{1}$. Note Γ is a multiplicative group.

<u>Theorem</u> A: If $z \in \mathrm{Ker}(\lambda)$, then there are representations U and V of G such that $r(z) = U - V$ and $U \underset{\mathbb{Q}}{\sim} V$. Here $r : R(G) \to RO(G)$ denotes "realification".

For cyclic groups with at least four distinct primes dividing $|G|$, Ker λ is non zero. In fact it's usually large. The main geometric ingredient in the proof of Theorem A is this theorem:

<u>Theorem</u> B: Let G be cyclic of odd order. Suppose W is a closed 4k dimensional framed manifold with G action such that

i) $\dim W^G = 0$

ii) For $H \subset G$, $H \neq 1$, the Euler characteristic of W^H $\chi(W^H)$ is 2 and $\dim W^H < \frac{1}{2} \dim W$

iii) $\mathrm{Sign}\,(G,W) = 0$

Then W is framed cobordant to W' rel$\{W^H | H \subset G, H \neq 1\}$ and W' is a rational homotopy sphere.

<u>Corollary C</u>: W^G consists of 2 points p and q and $T_p\Sigma \underset{\mathbb{Q}}{\sim} T_q\Sigma$.

<u>§ 2</u>. Outline of ideas used in theorems A and B.

We briefly indicate the ideas used in A) and B). This requires additional notation. Let Λ be $\mathbb{Z}$ or $\mathbb{Q}$ and n be an even integer. Let $W_n(G, \Lambda)$ be the equivariant Witt ring denoted by $W_n(\Lambda, G)$ in [ACH]. Briefly $W_n(G, \Lambda)$ consists of equivalence classes of pairs (M, ϕ) where M is a Λ torsion free $\Lambda(G)$ module and ϕ is a non singular, G invariant Λ valued bilinear form which satisfies $\phi(x, y) = (-1)^{n/2}\phi(y, x)$ for $x, y \in M$. If W is a closed manifold of dimension n with G acting preserving orientation, then $[W]_\Lambda \in W_n(G, \Lambda)$ is the class of $(H^{n/2}(W, \Lambda)/\text{Torsion}, \phi_W)$ where ϕ_W is the cup product bilinear form on W. We remark that $[W]_\Lambda$ depends only on the G cobordism class of W. Note this key observation:

2.1 $[W]_{\mathbb{Z}} = 0$ if W is a rational homology sphere. In the case $|G|$ is odd [ACH] give necessary and sufficient conditions that $[W]_{\mathbb{Z}} = 0$ which we exploit. To do this we <u>henceforth suppose</u> G <u>is an odd order cyclic group and</u> W <u>is a closed oriented smooth</u> G <u>manifold of dimension</u> $4k$ <u>and in addition we assume</u> $\dim W^G = 0$. In this case there is a simple formula for the torsion signatures $\{w_p(G, W) \mid p$ is a prime which divides $|G|\}$. Note that in the notation of [ACH] $w_p(G, W) = f(T, p)$ where T generates G. See [ACH] pages 149-151. Let p be a prime which divides $|G|$ and let P be

the p Sylow subgroup of G. Call p <u>good</u> if there is no integer x such that $-1 \equiv p^x \bmod |G/H|$; otherwise p is bad.

<u>Lemma 2.3</u>. Under the above assumptions on W, $w_p(G, W) = 0$ if p is good and $w_p(G, W) = \sum_{x \in W^G} \frac{1}{2}(\dim T_x W - \dim T_x W^P) \bmod 2$ if p is bad. (See 2.20)

Proof: This is immediate from [ACH, 1.8 p.141 and 3.5 p.149].

We emphasize that $w_p(G, W) \in \mathbb{Z}_2$ for each prime p which divides $|G|$. These invariants are all functions of $[W]_{\mathbb{Z}}$.

<u>Theorem 2.4</u>. [ACH, 3.6 p.151] $[W]_{\mathbb{Z}} = 0$ iff $\mathrm{Sign}(G, W) = 0$ and $w_p(G, W) = 0$ for all p which divide $|G|$.

<u>Corollary 2.5</u>. If W is a rational homology sphere with $W^G = x \cup y$ (2 points), then $\frac{1}{2}(\dim T_x W^P - \dim T_y W^P) \equiv 0(2)$ for each p Sylow subgroup for which p is bad.

Proof: This is immediate from 2.3 using the fact that $\dim T_x W$ and $\dim T_x W^P$ are even.

Corollary 2.5 gives an especially simple necessary condition that the representations U and V of G occur as $(T_x W, T_y W)$ for some smooth action of G on a rational homology sphere W with $W^G = x \cup y$. Actually much more stringent necessary conditions come from the condition $\mathrm{Sign}\,(G, W) = 0$. In fact if we add the condition that W be framed, all $w_p(G, W)$ vanish. Here is the argument:

<u>Theorem 2.6</u>. Suppose W is framed and $W^G = x \cup y$, then $w_p(G, W) = 0$ for all p which divide $|G|$.

Proof: The results of Atiyah in [A] assert:

$$+) \quad \mathrm{Ker}(R(G) \to K_G(E) = K(E/G) = \hat{R}(G))$$
$$= \mathrm{Ker}(R(G) \xrightarrow{\mathrm{res}} \prod_{P \text{ Sylow}} R(P))$$

(The P component of res is $(\mathrm{res})_P = \mathrm{res}_P$ where $\mathrm{res}_P : R(G) \to R(P)$ is restriction to $P \subset G$.) Clearly $T_xW - T_yW \in \mathrm{Ker}(R(G) \to K_G(E))$ if W is framed; so $T_xW - T_yW \in \mathrm{Ker}(R(G) \xrightarrow{\mathrm{res}} \prod_{P \text{ Sylow}} R(P))$. Now the assertion $w_p(G, W) = 0$ follows from 2.3. (See 2.19)

<u>Corollary 2.8</u>. Let W be a framed G manifold with $W^G = p \cup g$. Then $[W]_{\mathbb{Z}} = 0$ iff $\mathrm{Sign}(G, W) = 0$.

Now we discuss framed manifolds and equivariant surgery. The process of equivariant framed surgery is well understood when G acts freely on W. (See e.g. [W]). We treat this case first. Suppose G acts freely on W and $\beta : TW \cong (W \times V)$ is a stable G vector bundle isomorphism for some representation V of G. Call β a strong framing of W. Then for any $x \in \pi_j(W)$ $j \leq n/2$ $(n = \dim W)$, there is a G immersion (imbedding if $j < n/2$) $\iota : G \times S^j \times D^{n-j} \to W$ such that $\iota|S^j$ represents x. If ι is a G imbedding, there is a strong framing β' of $W' = W\text{-interior } (G \times S^j \times D^{n-j}) \cup G \times D^{j+1} \times S^{n-j-1}$ which agrees with β over $W\text{-interior } (G \times S^j \times D^{n-j})$. This construction $(W, \beta) \longmapsto (W', \beta')$ is called equivariant surgery and may be used to kill $\pi_j(W)$ for $j < n/2$. In fact W is

strongly framed cobordant to a manifold W'' with $\pi_j(W'') = 0$ for $j < n/2$. For elaboration of these ideas, see [PR].

This discussion generalizes as follows:

<u>Lemma 2.9</u>. Suppose W is framed and $\dim W^H < \frac{1}{2} \dim W$ whenever $H \neq 1$. Then W is framed cobordant $\mathrm{rel}\{W^H | H \neq 1\}$ to a manifold W'' with $\pi_j(W'') = 0$ for $j < n/2$. $(n = \dim W)$.

Proof: Here is an outline: Let $W^* = W - \bigcup_{H \neq 1} W^H$; so G acts freely on W^*. This means the projection of $E \times W^*$ on W^* is a G homotopy equivalence and this means that framing and strong framing of W^* is the same notion. Next note that the inclusion $W^* \to W$ induces an isomorphism in homotopy in dimensions not exceeding $n/2$; so any class $x \in \pi_j(W)$ $j < n/2$ comes from a class $x' \in \pi_j(W^*)$. Now note that the framing of W gives a framing of W^*; so W^* is strongly framed. Thus we may apply the above discussion to W^* and x'. This provides a G imbedding of $G \times S^j \times D^{n-j}$ in $W^* \subset W$, so we can form $W' = W\text{-interior } (G \times S^j \times D^{n-j}) \cup G \times D^{j+1} \times S^{n-j-1}$ as before. (Observe that $W'^H = W^H$ for all $H \neq 1$. This is the reason that the cobordism asserted is $\mathrm{rel}\{W^H | H \neq 1\}$.

Lemma <u>2.10</u>. Suppose $\chi(W^H) = 2$ for all $H \neq 1$ and $\tilde{H}_j(W, \mathbb{Q}) = 0$ for $j < n/2$ $n = \dim W$. Then $H_{n/2}(W, \mathbb{Q})$ and $H^{n/2}(W, \mathbb{Q})$ are free $\mathbb{Q}(G)$ modules.

<u>Proof</u>: By hypothesis $\dim W^G = 0$; so W^G is non empty. Let $x \in W^G$ and let V be the representation $T_x W$. Set $n=2k$ and $S = S(V \oplus \mathbb{R})$ where $\mathbb{R}$ is the trivial one dimensional real

representation and $S(V \oplus \mathbb{R})$ is the unit sphere of $V \oplus \mathbb{R}$. The Thom map $f : W \to S$ obtained by collapsing the exterior of an invariant disk centered at x has degree 1. Let M_f be the mapping cone of f. Then $\chi(M_f{}^H) = 1$ for $H \neq 1$ (because degree $f = 1$ and $\chi(W^H) = \chi(S^H) = 2$ for $H \neq 1$). In addition $\tilde{H}_i(M_f, \mathbb{Q}) = 0$ for $i \neq k + 1$. These two properties imply that $H_{k+1}(M_f, \mathbb{Q}) \cong H_k(W, \mathbb{Q})$ is a free $\mathbb{Q}(G)$ module. (See [O])

The obstruction to converting a framed manifold W satisfying:

2.11 $\quad \dim W^H < \frac{1}{2} \dim W$ and $\chi(W^H) = 2$ for $H \neq 1$.

into a rational homology sphere $\sum$ using equivariant surgery is an element $\sigma(W) \in L(\mathbb{Q}(G))$. Here $L(\mathbb{Q}(G))$ is an abbreviation for the Wall group $L_n^h(\mathbb{Q}(G), 1)$. Briefly this is an abelian group consisting of equivalence classes of triples (M, λ, μ) where M is a free $\mathbb{Q}(G)$ module, λ is a non singular, G invariant, $\mathbb{Q}$ valued bilinear form which satisfies $\lambda(x, y) = (-1)^{n/2}\lambda(y, x)$ for $x, y \in M$ and μ is an associated quadratic form. (See [W, §5] for notation). If W satisfies 2.11, it is framed cobordant to a manifold W'' which also satisfies 2.11 and in addition, $\pi_j(W'') = 0$ for $j < n/2$ (2.9). By 2.10 $M = H^{n/2}(W'', \mathbb{Q})$ is a free $\mathbb{Q}(G)$ module. Then $\sigma(W)$ is the class of $(H^{n/2}(W'', \mathbb{Q}), \phi_{W''}, \mu_{W''})$ where $\mu_{W''}$ is the self intersection form of W'' (See [W, §5]).

There is an obvious homomorphism $\rho : L(\mathbb{Q}(G)) \to W_n(G, \mathbb{Q})$ which sends $\sigma(W)$ to $[W]_{\mathbb{Q}}$. Because n is $0 \bmod 4$, ρ is

injective. We can now give a proof of Theorem B.

<u>Proof of Theorem B</u>: By i) and ii), W^G consists of two points x and y. By 2.8 $[W]_{\mathbb{Z}} = 0$ and this implies that $[W]_{\mathbb{Q}} = 0$. But $[W]_{\mathbb{Q}} = \rho\sigma(W)$. Since ρ is injective, $\sigma(W) = 0$. Since $\sigma(W)$ is the obstruction to converting W to a rational homology sphere $\sum$ and since $\sigma(W) = 0$, $\sum$ exists.

Now we turn to the discussion of Theorem A. We view the cyclic group G as the subgroup of $\mathbb{C}^x = \mathbb{C} - 0$ consisting of the $|G|$th roots of unity. Let t^i denote the complex one dimensional representation of G on which $g \in G$ acts on $v \in t^i$ by $g(v) = g^i \cdot v$ i.e. complex multiplication by g^i. A complex representation V of G may be uniquely written as $V = \sum_{i=0}^{|G|-1} a_i t^i$ for some integers $a_i \geq 0$. For $g \in G$, $V^g = \{v \in V | gv = v\}$. When $V^g = 0$, we can define this complex number:

$$\nu(V)(g) = \prod_{i=0}^{|G|-1} \left(\frac{1+g^i}{1-g^i}\right)^{a_i} \in \mathbb{C}^x. \tag{2.12}$$

The assumption $V^g = 0$ means the denominator does not vanish. These complex numbers appear in the Atiyah Singer index formula for $\mathrm{Sign}(g, W)$ when $\dim W^g = 0$. Here is a discussion of this point. Suppose $W^g = W^G$. (By hypothesis $\dim W^G = 0$.) Let $x \in W^G$. Since G preserves orientation, there is complex representation of G whose realification is T_xW. Choose one $\widetilde{T_xW}$ for which the orientation given by the complex structure agrees with the given orientation on T_xW. Then

2.13 $$\mathrm{Sign}(g, W) = \sum_{x \in W^G} \nu(\widetilde{T_x W})(g).$$

We remark that if V and V' are two complex representations whose realifications are both $T_x W$, then $\nu(V')(g) = \pm\nu(V)(g)$; so there is a sign ambiguity for the right hand side of 2.13 as a function of the <u>real</u> representation $T_x W$. This is ambiguity disappears when orientation is accounted for in the way mentioned. Another relevant elementary point is that if $r(V) = T_x W$, there is a complex representation V' such that $r(V') = r(V)$ and $\nu(V')(g) = -\nu(V)(g)$ for all g for which $\dim V^g = 0$.

Theorem B is used in the proof of Theorem A. To use Theorem B for this purpose we need to produce a framed manifold W with $W^G = x \cup y$ (two points) and $\mathrm{Sign}(G, W) = 0$. Let $V = \widetilde{T_x W}$ and $U = \widetilde{T_y W}$ and let g be an element of G for which $W^g = 0$. Then $V^g = 0 = V^g$ and

$$\mathrm{Sign}(g, W) = \nu(V)(g) + \nu(U)(g).$$

so

2.14 $$0 = \mathrm{Sign}(g, W) \quad \text{iff} \quad \nu(V)(g)/\nu(U)(g) = -1.$$

In summary we have obtained these conditions on two representations U and V of G:

<u>Lemma 2.15</u>. Let W be a framed G manifold with $W^G = x \cup y$, $\widetilde{T_x W} = V$, $\widetilde{T_y W} = U$ and $\mathrm{Sign}(G, W) = 0$. Then

(i) $V - U \in \mathrm{Ker}(R(G) \xrightarrow{\mathrm{res}} \prod_{P \text{ Sylow}} R(P))$ and

(ii) $\nu(V)(g)/\nu(U)(g) = -1$ whenever $W^g = W^G$.

(Note this implies $U^g = V^g = 0$.)

Lemma 2.15 and the discussion preceeding it lead to sufficient conditions that two representations U and V occur (stably) as $T_x\Sigma$ and $T_y\Sigma$ for some rational homology sphere Σ with $\Sigma^G = x \cup y$. We discuss this point.

Let S_1 be the set of divisors d of $|G|$ such that $|G|/d$ is a prime power and let S_2 be the set of divisors d of $|G|$ such that $|G|/d$ is divisible by at most three distinct primes. Let

2.16 $\quad \overline{R}(G) = \{U - V \in R(G)$ such that i-iii hold$\}$.

i) $\quad V^g = U^g = 0$ whenever $g \in G$ and $|g| \in S_1$

ii) $\quad \dim V^g = \dim U^g$ whenever $|g| \in S_2$

iii) $\quad V - U \in \mathrm{Ker}(R(G) \xrightarrow{\text{res}} \prod_{P \text{ Sylow}} R(P))$

We define the homomorphism λ in Theorem A.

$$\lambda : \overline{R}(G) \longrightarrow \prod_{S_1} \mathbf{C}^x/\mathbf{Z}_2.$$

If $d \in S_1$, the d th coordinate of λ is

$$\lambda_d(V - U) = \nu(V)(g)/\nu(V)(g) \qquad g = \exp(2\Pi i/d) \in G.$$

We can only very briefly discuss the points of the proof of Theorem A. If $z \in \mathrm{Ker}\ \lambda$, there is a manifold W satisfying the assumptions of Theorem B and in addition $W^G = x \cup y$ and $T_xW - T_yW = r(z)$. Theorem A follows from this and Corollary C. Here are the essential points: There are complex representations U and V of G satisfying 2.16 i-iii and in

addition

2.17 i) $r(V - U) = r(z)$

ii) $\lambda(V - U) = -\underline{1}$.

(If $\lambda(z) = -1$, $V - U = z$. If $\lambda(z) = 1$, then $V - U$ is not z, but $r(V - U) = r(z)$. This is related to the discussion after 2.13.) Use 2.16 i) and the methods of [P] to produce manifolds $X(V)$ and $X(U)$ with these properties:

2.18 i) $X(V)^G = x$(one point), $X(U) = y$ (one point)

ii) $X(V)^g = X(V)^G$ and $X(U)^g = X(U)^G$ whenever $|g| \in S_1$.

iii) $TX(V)$ and $TX(U)$ are stably G isomorphic to $X(V) \times V$ and $X(U) \times U$ respectively.

By 2.16 iii) and 2.6 +), $W = X(V) \amalg X(U)$ is framed; moreover, $X^g = x \cup y$ whenever $|g| \in S_1$ and $\widetilde{T_x W} = V$, $\widetilde{T_y W} = U$ by construction. (Note 2.18 iii) which implies $T_x W = r(V)$ and $T_y W = r(U)$.) Whenever $|g| \in S_1$, $\mathrm{Sign}(g, W) = 0$ because $\lambda_{|g|}(V - U) = \nu(V)(g)/\nu(U)(g) = -1$. See 2.14. Of course the condition $\mathrm{Sign}(G, W) = 0$ requires $\mathrm{Sign}(g, W) = 0$ for all $g \in G$ not just $g \in G$ with $|g| \in S_1$. The fact that $\mathrm{Sign}(g, W) = 0$ for $|g| \notin S_1$ and the other properties of W required for Theorem B are consequences of other properties of the construction of W which we omit.

2.19 At some points in text we do not distinguish between real and complex representations. Since G is cyclic this should cause no problem.

2.20 The assumption $\dim W = 4k$ may be dropped.

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