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Characterization of semipolar sets for Lévy processes by Fourier transform of measures

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In this note we study certain characterization of semipolar sets for d-dimensional Lévy processes with density. First we explain the fact below:

> A set is nonpolar and semipolar if and only if there is a gap between the usual energy integral of measures supported by the set and its Fourier transform virsion:.

This type of result was first explicitely stated by Rao[6] without proof. Our result is a slight refinement of Rao's result and an application is given. Here we note that no concrete examples of nonpolar semipolar sets of our processes are known except the case that a point is nonpolar. Unfortunately, even now, we do not know whether there exists another type of nonpolar semipolar sets for our class of Lévy processes. So the value of some of our results is not clear, for which we give only an outline of the proof.

1. Lévy processes

Let X_t be a Lévy process on R^d (i.e. a process with stationary independent increments). The Lévy-Hincin formula states that the characteristic function of X_t-X_0 is given by

$$E\exp[i(z,X_t-X_0)] = \exp[-t\Psi(z)],$$

where $\Psi(z)$, the exponent, is given by

$$\Psi(z) = i(a,z) + Q(z) + \int \{1-\exp[i(z,y)]+i(z,y)\ (1+|y|^2)^{-1}\}n(dy).$$

Here a is a constant vector, Q is a non-negative definite quadratic form, and $n(dy)$ is the so-called Lévy measure and is such that $\int |y|^2 (1+|y|^2)^{-1} n(dy) < \infty$. If $\lambda>0$, the potential kernel is defined by

$$U^\lambda(x,A) = \int_0^\infty \exp(-\lambda t) P(t,x,A)\,dt, \quad P(t,x,A) = P(X_t \in A \mid X_0 = x),$$

then

$$\int \exp[i(z,y)] U^\lambda(0,dy) = [\lambda + \Psi(z)]^{-1}.$$

In §2 we assume that for each x

A_1) $A \mapsto U^\lambda(x,A)$ is absolutely continuous with respect to Lebesgue measure.

Then

$$U^\lambda(x,A) = \int_A u^\lambda(y-x)\,dy$$

and we can choose u^λ so that $x \mapsto u^\lambda(y-x)$ is λ-excessive for each fixed y and lower semicontinuous. Further $U^\lambda f(x) = \int u^\lambda(y-x) f(y)\,dy$ $(\lambda>0)$ maps the class of bounded measurable functions into the class of continuous functions. In §3 we pose the stronger condition:

A_2) $P(t,x,A)$ has a bounded continuous density with respect to Lebesgue measure for each fixed $t>0$.

Under the condition A_1), there exists a unique measure μ_K^λ with support in $\bar{K}$ such that $E_x(\exp[-\lambda\sigma_K]) = \int u^\lambda(y-x)\mu_K^\lambda(dy)$, where $\sigma_K = \inf(t>0, X_t \in K)$ for every Borel set K. (Blumenthal-Getoor[1]). Setting

$$C^\lambda(K) = \mu_K^\lambda(\bar{K}),$$

we call it the λ-capacity of K. If $C^\lambda(K) = 0$, the set K is

said to be polar. Set $K^r = \{x;\ P_x(\sigma_K=0) = 1\}$ and denote the class of semipolar sets by SP , that is, $SP = \{ K;\ K \subset \bigcup_{\nu=1}^{\nu} K_i,\ K_i^r = \phi \}$. (We omit the measurability condition in the above.)

For a function f or a measure μ , we denote the Fourier transform by

$$\hat{f}(z)\ (\ \hat{\mu}(z)\ ) = \int \exp[i(z,y)]f(y)dy \ (\text{resp. } \int \exp[i(z,y)]\mu(dy),$$

if well defined. The symbol $*$ is used for the convolution.

2. Bochner's theorem and Rao's theorem on the energy integral

Throughout this section we pose the condition A_1). For each signed measure μ we set

$$I^\lambda(\mu) = \iint \mu(dy)u^\lambda(y-x)\mu(dx),$$

if defied, and call it the energy integral of μ . Set

$$\mu_s = \mu * \tilde{\mu}, \quad u_S^\lambda(x) = 2^{-1}(\ u^\lambda(x) + u^\lambda(-x)\ ),$$

where $\tilde{\mu}(A)=\mu(-A)$. Then

$$I^\lambda(\mu) = u_S^\lambda * \mu_S(0).$$

The well-known Bochner's theorem ensures

<u>Theorem</u>. (Bochner[2], Theorem2.1.5 and Theorem 2.2.1.)

i) If $f \in L^1$ and $\hat{f} \in L^1$, then $f(x) = (2\pi)^{-d}\int \exp[-i(x,z)]\hat{f}(z)dz$ for almost all x .

ii) If $f \in L^1$, f;bounded, and $\hat{f} \geq 0$, then $\hat{f} \in L^1$.

Let μ be a bounded signed measure such that $u_S^\lambda * \mu_S$ is bounded. Since $\widehat{u_S^\lambda * \mu_S}(z) = \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\mu(z)|^2 \geq 0$, it follows from Bochner's theorem that

$$u_S^\lambda * \mu_S(x) = (2\pi)^{-d}\int \exp[i(x,z)]\mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}(z)|^2 dz$$

for almost all x. Since the right term is continuous and the left

term is lower semicontinuous if μ is a non-negative measure, we have

$$I^{\lambda}(\mu) \leq (2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}(z)|^2 dz$$

for each non-negative measure such that $u_S^{\lambda}*\mu$ is bounded.

Rao states the following

Theorem (M.Rao[6]) Every semipolar set is polar if and only if

$$I^{\lambda}(\mu) = (2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}(z)|^2 dz$$

for every bounded signed measure μ such that $u_S^{\lambda}*\mu$ is bounded.

Instead of giving the proof we show here that his theorem is true even if we replace signed measures with non-negative measures.

Theorem 1. Every semipolar set is polar if and only if the equality in Rao's theorem holds for each non-negative bounded measure μ such that $u_S^{\lambda}\mu$ is bounded, or equivalently

$$\lim_{\lambda\uparrow\infty}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}(z)|^2 dz = 0,$$

for such measure μ.

The "only if" part is contained in Rao's theorem. For the proof of "if" part and for the later use we prepare a lemma.

Lemma 1. Let μ be a bounded signed measure such that $u_S^{\lambda}*\mu$ is bounded. If $\mathrm{Re}([\lambda+\Psi(z)]^{-1})\hat{\mu}(z)\in L^1$, then

$$u_S^{\lambda}*\mu(0) + \lim_{\lambda\uparrow\infty}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})\hat{\mu}(z)dz = \int \mathrm{Re}([\lambda+\Psi(z)]^{-1})\hat{\mu}(z)dz.$$

Especially, if μ is a bounded non-negative measure such that $u_S^{\lambda}*\mu$ is bounded, then

$$I^{\lambda}(\mu) + \lim_{\lambda\uparrow\infty}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}(z)|^2 dz = \int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}(z)|^2 dz$$

Proof. Since $u_S^{\lambda}*\mu$ and $\mathrm{Re}([\lambda+\Psi(z)]^{-1})\hat{\mu}(z)$ belong to L^1, it follows from i) of Bochner's theorem that

$$u_S^\lambda * \mu(x) = (2\pi)^{-d}\int e^{-i(x,z)} \mathrm{Re}([\lambda+\Psi(z)]^{-1})\hat{\mu}(z)dz \quad \text{for almost all} \quad x .$$

By the resolvent equation we have, for ,

$$u^\lambda * \mu - u^{\lambda'} * \mu = (\lambda'-\lambda)u^\lambda * u^{\lambda'} * \mu , \quad \tilde{u}^\lambda * \mu - \tilde{u}^{\lambda'} * \mu = (\lambda'-\lambda)\tilde{u}^\lambda * \tilde{u}^\lambda \mu .$$

Since u^λ and $\tilde{u}^\lambda$ map the class of all bounded measurable functions into the class of continuous functions, $u_S^\lambda * \mu - u_S^\lambda * \mu$ is continuous. Hence

$$u_S^\lambda * \mu(x) - u_S^\lambda * \mu(x) = (2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1}\exp[-i(x,z)]\hat{\mu}(z)dz$$
$$- (2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1}\exp[-i(x,z)]\hat{\mu}(z)dz$$

everywhere. Letting $\lambda\uparrow\infty$ at $x=0$, we get the desired equality. For the proof of the latter half we have only to consider $\mu*\tilde{\mu}$ instead of μ and use ii) of Bochner's theorem.

3. Refinement of Theorem 1.

In this section we give a refinement of Theorem 1. We do not use Rao's theorem, but we need the previous result [4] and so we assume the condition A_2) throughout this section.

<u>Lemma 2</u>. If there exists a nonpolar semipolar set, we can find a compact subset K such that $0<\lim_{\lambda\uparrow\infty}C^\lambda(K)<\infty$, and further if we set $\mu_K^\infty(B)=\lim_{\lambda\uparrow\infty}C^\lambda(B)$ for each Borel subset of K, μ_K^∞ defies a non-negative measure on K .

The first half is proved in [4]. We omit the latter half. Instead, we note that above lemma can be refined as follows;

there exists a non-negative but non-σ-finite measure π on the Borel field $\mathcal{B}$ so that i) $B\in\mathcal{B}$ is semipolar if and only if the restriction $\pi|_B$ of π to B is σ-finite, ii) $B\in\mathcal{B}$ is polar if and only if $\pi(B)=o$, iii) for a bounded Borel set B , $\pi(B)<\infty$ if and only if $\lim_{\lambda\uparrow\infty}C^\lambda(B)<\infty$ and in this case $\pi(B)=\lim_{\lambda\uparrow\infty}C^\lambda(B)$.

Theorem 2. Let μ be a non-negative bounded measure such that $u_S^\lambda * \mu$ is bounded. If μ carries a semipolar set, we can choose a compact subset K of the set such that $\mu(K)>0$ and

$$\lim_{\lambda\uparrow\infty}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}|_K(z)|^2dz = \int (d\mu|_K/d\mu_K^\infty(z))^2 d\mu_K^\infty(z),$$

where $\mu|_K$ is the restriction of μ to K and μ_K^∞ is the measure defined by Lemma 2. Especially

$$\lim_{\lambda\uparrow\infty}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}_K^\infty(z)|^2dz = \mu_K^\infty(K)$$

for the compact set K chosen in the above.

Proof. For simplicity we pove the latter half. By Lemma 2 we can choose a compact set K of a semipolar set carried by μ so that μ_K^∞ is a non-negative bounded measure on K. Further, using the strong Markov property, we can rechoose K so that

$$(*)\quad \lim_{\lambda\uparrow\infty}[(\sup_{R^d} u^\lambda * \mu_K^\infty)-1] = \lim_{\lambda\uparrow\infty}[(\sup_{R^d} \tilde{u}^\lambda * \mu_K^\infty)-1] = 0.$$

Choose an open neighborhood Q of K. Then Hawkes's theorem [3] ensures

$$C^\lambda(Q)^{-1} = \inf\{(2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\nu}(z)|^2dz,\ \nu\in Pr(Q)\},$$

where $Pr(Q)$ is the class of probability measures whose support is in Q. Hence

$$C^\lambda(Q)^{-1} \leqq (\mu_K^\infty(K))^{-2}(2\pi)^{-d}\int |\hat{\mu}_K^\infty(z)|^2\mathrm{Re}([\lambda+\Psi(z)]^{-1})dz.$$

Therefore

$$(2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\mu_K^\infty(z)|^2dz \geqq \mu_K^\infty(K)\mu_K^\infty(K)/C^\lambda(Q).$$

Since we can choose $Q\downarrow K$ so that $C^\lambda(Q)\downarrow C^\lambda(K)$ and $\mu_K^\infty(K)\geqq C^\lambda(K)$, we have

$$(2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}_K^{\infty}(z)|^2 dz \gneqq \mu_K^{\infty}(K).$$

In the next choose the mollifier ρ_ε and denote by μ_ε the measure whose density is $\mu_K^{\infty}*\rho_\varepsilon$. Then

$$(2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}_K^{\infty}(z)|^2|\hat{\rho}_\varepsilon(z)|^2 dz = I^{\lambda}(\mu_\varepsilon).$$

Since the left terms tend to $(2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)]^{-d}|\mu_K^{\infty}(z)|^2 dz$ as ε tends to zero and $I^{\lambda}(\mu) \leqq \sup_{\mathbb{R}^d} u_S^{\lambda}*\mu_K^{\infty}\cdot\mu_K^{\infty}(K)$, it follows from (*) that

$$\lim_{\lambda\uparrow\infty}(2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}_K^{\infty}(z)|^2 dz \leqq \mu_K^{\infty}(K).$$

We have finished the proof of the latter half of the theorem.

As a simple consequence of Theorem 2 we can get a kind of localization of Theorem 1 as follows.

Remark 1. Let μ be a non-negative bounded measure such that $u_S^{\lambda}*\mu$ is bounded. The measure μ does not carry a semipolar set if and only if

$$I^{\lambda}(\mu) = (2\pi)^{-d}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}(z)|^2 dz,$$

or equivalently

$$\lim_{\lambda\uparrow\infty}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})|\hat{\mu}(z)|^2 dz = 0.$$

The non-negative measure π introduced just after Lemma 2 would be basic, since

Remark 2. Let μ be a non-negative bounded measure such that $u_S^{\lambda}*\mu$ is bounded. Then μ is absolutely continuous with respect to π.

From the proof of Theorem 2 we also have

Remark 3. If the energy integral $I^{\lambda}(\mu)$ is vaguely continuous, every semipolar set is polar.

4. One-dimensional case.

For one-dimensional Lévy processes we can modify our previous results to those which seem easy for applications. In this section we always assume that

Every point is polar.

First we give another virsion of Theorem 1.

Theorem 3. Assume A_1). Then every semipolar set is polar if and only if

$$\lim_{\alpha\downarrow 0}\int \mathrm{Re}([\lambda+\Psi(z)]^{-1})*V_\alpha(z)|\hat{\mu}(z)|^2 dz = 0$$

for every non-negative measure μ of compact support such that $u_S^\lambda*\mu$ is bounded and a fixed $\lambda>0$, where

$$V_\alpha(z) = 2K_{2\alpha}(z) - K_\alpha(z),$$

$$K_\alpha(z) = (\alpha/2\pi)(\sin(\alpha z/2)/(\alpha z/2))^2;\ \text{the Fejer kernel.}$$

Using this theorem we can show that every semipolar set is polar for the processes treated in [5]. The proof is omitted but not so trivial, since we use some technical results in [5].

Corollary 1. Let us assume A_1) and suppose that, for a fixed $\lambda>0$ there exist α $(1>\alpha>0)$, and a continuous function F on $(0,\infty)$ such that $M_1F(z)\leq \mathrm{Re}([\lambda+\Psi(z)]^{-1})\leq M_2F(z)$ for every large z, where $0<M_1\leq M_2<\infty$, and $z^\alpha F(z)$ is decreasing on on $(0,\infty)$, and $\mathrm{Re}([\lambda+\Psi(2z)]^{-1})/\mathrm{Re}([\lambda+\Psi(z)]^{-1}) > M>0$ for every $z>0$. Then every semipolar set is polar.

We can also show the following by Theorem 3.

Corollary 2. Suppose that given two Lévy processes X_1 and X_2 satisfy A_1). Let Ψ_1 and Ψ_2 be the exponents respectively. If

$$M_1\mathrm{Re}([\lambda+\Psi_1(z)]^{-1}) \leq \mathrm{Re}([\lambda+\Psi_2(z)]^{-1})\leq M_2\mathrm{Re}([\lambda+\Psi_1(z)]^{-1})$$

holds for every z and some fixed $\lambda>0$, where $0<M_1<M_2<\infty$,

then the classes of semipolar sets for X_1 and X_2 coincide.

The above result also follows from more general result.

The next result is proved using Theorem 3 together with Wiener's result (Theorem 21, [7]).

Theorem 4. Assume A_1). Then every semipolar set is polar if and only if

$$\lim_{T\uparrow\infty} T^{-1}\int_0^T [\int \mathrm{Re}([\lambda+\Psi(z-x)]^{-1})|\hat{\mu}(z)|^2 dz]dx = 0$$

for every non-negative measure μ of compact support such that $u_S^\lambda * \mu$ is bounded.

The following result seems to be trivial. However the proof we get at present needs both Theorem 2 and Theorem 4, and so we have to put the condition A_2). It would be proved more easily and in more general form by probabilistic method.

Theorem 5. Assume A_2). Let μ be a non-negative measure such that $u_S^\lambda * \mu$ is bounded. Then $\mu * \tilde{\mu}$ does not carry a semipolar set.

We did not give proofs for all the results above, aithough they are not so trivial. This is because the author does not abandon an optimistic conjecture: every semipolar set is polar for one-dimensional Lévy processes with A_1) for which every point is polar.

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