

THE ROLE OF BOUNDARY HARNACK PRINCIPLE

IN THE STUDY OF PICARD PRINCIPLE

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A nonnegative locally Hölder continuous function  $P$  on  $0 < |z| \leq 1$  will be referred to as a *density* on  $\Omega: 0 < |z| < 1$ . A density on  $\Omega$  gives rise to an elliptic operator  $L_P$  on  $\Omega$  defined by

$$(1) \quad L_P u = \Delta u - Pu, \quad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2.$$

We say that the *Picard principle* (abbreviated as PP) is valid for  $P$ , rather for  $L_P$ , at  $z=0$  if the dimension of the half module of nonnegative solutions of  $L_P u = 0$  on  $\Omega$  with vanishing boundary values on  $\partial\Omega - \{z=0\}$  is 1. With the operator  $L_P$  we associate an elliptic operator  $\hat{L}_P$  on  $\Omega$ , referred to as the *associate operator* to  $L_P$ , given by

$$(2) \quad \hat{L}_P v = \Delta v + 2V \log e_P \cdot \nabla v, \quad \nabla = (\partial/\partial x, \partial/\partial y),$$

where  $e_P$ , referred to as the *P-unit* on  $\Omega$ , is the unique bounded solution of  $L_P u = 0$  on  $\Omega$  with boundary values 1 on  $\partial\Omega - \{z=0\}$ . We also say that the *Riemann theorem* (abbreviated as RT) is valid for  $\hat{L}_P$  at  $z=0$  if the limit  $\lim_{z \rightarrow 0} v(z)$  exists for every bounded solution  $v$  of  $\hat{L}_P v = 0$  on  $\Omega$ . Then we have the *duality theorem* (cf. Heins [3], Hayashi [2], Nakai [8]): The Picard principle is valid for  $L_P$  at  $z=0$  if and only if the Riemann theorem is valid for  $\hat{L}_P$  at  $z=0$ . As a sufficient condition for the Riemann theorem for  $\hat{L}_P$  at  $z=0$  we have, what we call, the *boundary Harnack principle* (abbreviated as BHP) for  $L_P$  at  $z=0$  (Kawamura [6]):

$$(3) \quad \left\{ \begin{array}{l} \text{For every Jordan region } U \text{ in } |z| < 1 \text{ containing } z=0 \text{ there exists a} \\ \text{Jordan region } V_U \text{ containing } z=0 \text{ such that } \bar{V}_U \subset U \text{ and } u(\zeta) \leq C u(\xi) \text{ for} \\ \text{every nonnegative bounded solution } u \text{ of } L_P u = 0 \text{ on } U - \{z=0\} \text{ and } \zeta, \xi \\ \text{in } \partial V_U, \text{ where } C \text{ is a positive constant independent of } U, u, \zeta, \text{ and } \xi. \end{array} \right.$$

In fact (3) implies the *boundary Harnack principle* for  $\hat{L}_p$  at  $z=0$  which is formulated in the same fashion as it is done for  $L_p$  originally considered by Kawamura [6] and then the Riemann theorem for  $\hat{L}_p$  at  $z=0$  is deduced from the boundary Harnack principle for  $\hat{L}_p$  at  $z=0$  ([6]). In short it has been known that the following string of implications holds:

$$(4) \quad \text{BHP for } L_p \implies \text{BHP for } \hat{L}_p \implies \text{RT for } \hat{L}_p \iff \text{PP for } L_p.$$

The purpose of this lecture is to show that the Picard principle for  $L_p$  conversely implies the boundary Harnack principle for  $L_p$  ([9]). Therefore we can conclude that properties appearing in (4) are in fact all equivalent to each other :

$$(5) \quad \text{BHP for } L_p \iff \text{BHP for } \hat{L}_p \iff \text{RT for } \hat{L}_p \iff \text{PP for } L_p.$$

We will also give an example of a density satisfying the boundary Harnack principle at  $z=0$  ([9]): If a density  $P$  on  $\Omega$  satisfies  $Q(z) \leq P(z) \leq Q(z) + C/|z|^2$  for a positive constant  $C$  and a *rotation free* density  $Q$  on  $\Omega$ , i.e. a density satisfying  $Q(z) = Q(|z|)$ , for which the Picard principle is valid at  $z=0$ , then the boundary Harnack principle is valid for  $L_p$  at  $z=0$  so that the Picard principle is valid for  $L_p$  at  $z=0$ .

### 1. The Harnack principle.

We will define a Harnack constant  $C(K, \Omega_a, P)$  and deduce the *ordinary* Harnack principle. For a density  $P$  on  $\Omega$  and a real number  $a$  in  $(0,1]$  we denote by  $G_p^{\Omega_a}$  the  $P$ -Green's function on  $\Omega_a = \{0 < |z| < a\}$ , i.e. the Green's function on  $\Omega_a$  with respect to the equation  $L_p u = 0$ . We consider a Harnack constant  $C(K, \Omega_a, P)$  of a compact subset  $K$  of  $\Omega_a$  defined by

$$C(K, \Omega_a, P) = \max \left\{ \frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \xi)} ; |z| = a \text{ and } \zeta, \xi \text{ are in } K \right\},$$

where  $\partial/\partial n_z$  means the inner normal derivative. Then the integral representation of a bounded solution of  $L_P u = 0$  in terms of the inner normal derivative of the P-Green's function yields the following Harnack principle: for any nonnegative bounded solution  $u$  of  $L_P u = 0$  on  $\bar{\Omega}_a - \{z=0\}$  and  $\zeta, \xi$  in  $K$  we have

$$u(\zeta) \leq C(K, \Omega_a, P)u(\xi).$$

## 2. The boundary Harnack principle.

We will show that the Picard principle for  $L_P$  implies the boundary Harnack principle for  $L_P$ , and hence they are equivalent. Let  $P$  be a density on  $\Omega$  such that the Picard principle is valid for  $L_P$  at  $z=0$ . Then the function  $G_P^{\Omega_a}(z, \zeta)/e_P(\zeta)$  in  $z$  converges uniformly on every compact subset of  $\bar{\Omega}_a - \{z=0\}$  as  $\zeta \rightarrow 0$ , and hence the inner normal derivative  $\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)/e_P(\zeta)$  converges to a positive continuous function on  $\partial\Omega_a - \{z=0\}$  (cf Itô [5]). In order to show (3) we consider two cases separately:  $\limsup_{\zeta \rightarrow 0} e_P(\zeta) = 0$  and  $> 0$ .

First we consider the case  $\limsup_{\zeta \rightarrow 0} e_P(\zeta) = 0$ , i.e.  $\lim_{\zeta \rightarrow 0} e_P(\zeta) = 0$ . For every  $\lambda$  in  $(0, 1)$  let  $A_\lambda$  be a connected component of  $\{\zeta \in \Omega ; e_P(\zeta) < \lambda\}$  such that  $z=0$  is an isolated boundary point of  $A_\lambda$ . Observe that  $\bar{A}_\lambda \downarrow \{z=0\}$  as  $\lambda \rightarrow 0$  and

$$\frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \xi)} = \frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)}{e_P(\zeta)} \frac{e_P(\xi)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \xi)}$$

for  $\zeta, \xi$  in  $\partial A_\lambda - \{z=0\}$ . Then we have

$$\lim_{\lambda \rightarrow 0} C(\partial A_\lambda - \{z=0\}, \Omega_a, P) = 1$$

so that for every subregion  $U$  of  $\{|z| < 1\}$  containing  $z=0$  we can take  $a_U, \lambda_U$  in  $(0,1)$  with  $\Omega_{a_U} \subset U$  and  $C(\partial A_{\lambda_U} - \{z=0\}, \Omega_{a_U}, P) < 2$ . Therefore (3) is valid for  $C = 2$  and  $V_U = A_{\lambda_U} \cup \{z=0\}$ .

Assume next that  $\limsup_{\zeta \rightarrow 0} e_P(\zeta) \equiv \delta > 0$ . There exists a closed set  $E$  thin at  $z=0$  in  $\Omega$  such that  $e_P(\zeta) \rightarrow \delta$  as  $\zeta \rightarrow 0$  with  $\zeta \notin E$  (cf BreLOT [1]). Then we can take a decreasing sequence  $\{\lambda_n\}_1^\infty$  in  $(0,1)$  with  $E \cap U_1^\infty(\partial \Omega_{\lambda_n} - \{z=0\}) = \emptyset$  and  $\lim \lambda_n = 0$ . Observe that  $e_P(\zeta) \rightarrow \delta$  as  $\zeta \rightarrow 0$  with  $\zeta \in U_1^\infty(\partial \Omega_{\lambda_n} - \{z=0\})$  and

$$\frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \xi)} = \frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)}{e_P(\zeta)} \frac{e_P(\xi)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \xi)} \frac{e_P(\zeta)}{e_P(\xi)}$$

for  $\zeta, \xi$  in  $\partial \Omega_{\lambda_n} - \{z=0\}$ . Then we have

$$\lim_{n \rightarrow \infty} C(\partial \Omega_{\lambda_n} - \{z=0\}, \Omega_a, P) = 1.$$

Therefore (3) is valid for  $C = 2$  and  $V_U = \Omega_{\lambda_n} \cup \{z=0\}$  for some  $n$  depending on  $U$ .

### 3. Fundamental properties of units.

We now recall some of fundamental properties of the  $Q_n$ -unit. Let  $Q$  be a *rotation free* density on  $\Omega$ , i.e. a density satisfying  $Q(z) = Q(|z|)$ . We consider a rotation free density  $Q_n(z) = Q(z) + n^2/|z|^2$  on  $\Omega$  for every nonnegative integer  $n$  and the  $Q_n$ -unit  $f_n(z, a)$  on  $\Omega_a$ , i.e. an unique bounded solution of  $L_{Q_n} u = 0$  on  $\Omega_a$  with boundary values 1 on  $\partial \Omega_a - \{z=0\}$ , where we follow the convention  $Q_0 = Q$  and  $f_0(z, 1) = e_Q(z)$ . Then  $f_n(z, a)$  is rotation free and  $f_n(r, a)$  is an unique bounded solution of

$$(6) \quad \ell_n \psi(r) \equiv \ell_{Q_n} \psi(r) \equiv \frac{d^2}{dr^2} \psi(r) + \frac{1}{r} \frac{d}{dr} \psi(r) - Q_n(r) \psi(r) = 0$$

on  $(0, a)$  with boundary values 1 at  $r = a$ . We have the following properties of  $f_n(r, a)$  (cf Nakai [7]):

$$(7) \quad f_n(r, \rho) = \frac{f_n(r, a)}{f_n(\rho, a)} \quad (0 < r \leq a, r \leq \rho \leq a);$$

$$(8) \quad f_n(r, a) > f_{n+1}(r, a) \quad (0 < r < a);$$

$$(9) \quad \frac{f_{n+1}(r, a)}{f_n(r, a)} \geq \frac{f_{n+2}(r, a)}{f_{n+1}(r, a)} \quad (0 < r \leq a);$$

$$(10) \quad \left\{ \frac{f_{n+1}(r, a)}{f_n(r, a)} \right\}^3 \leq \frac{f_{n+2}(r, a)}{f_{n+1}(r, a)} \quad (0 < r \leq a);$$

the Picard principle is valid for  $L_Q$  at  $z = 0$  if and only if

$$(11) \quad \lim_{r \rightarrow 0} \frac{f_1(r, a)}{f_0(r, a)} = 0$$

for some  $a$ , and hence by (7) any  $a$  in  $(0, 1]$ . For another rotation free density  $R$  on  $\Omega$  with  $Q \leq R$  we have also (cf Imai [4])

$$(12) \quad \frac{f_{n+1}(r, a)}{f_n(r, a)} \leq \frac{g_{n+1}(r, a)}{g_n(r, a)} \quad (0 < r \leq a),$$

where  $R_n(z) = R(z) + n^2/|z|^2$  and  $g_n(z, a)$  is the  $R_n$ -unit on  $\Omega_a$  ( $n=0, 1, \dots$ ).

#### 4. Fourier coefficients of solutions.

We consider Fourier coefficients

$$\begin{cases} c_0(r, w) = \frac{1}{2\pi} \int_0^{2\pi} w(re^{i\theta}) d\theta, \\ a_n(r, w) = \frac{1}{\pi} \int_0^{2\pi} w(re^{i\theta}) \cos n\theta d\theta, \end{cases}$$

$$b_n(r, w) = \frac{1}{\pi} \int_0^{2\pi} w(re^{i\theta}) \sin n\theta d\theta$$

for a continuous function  $w(z)$  on  $\bar{\Omega}_a - \{z=0\}$ . Here and hereafter let  $Q$  be a rotation free density on  $\Omega$  and  $f_n(z, a)$  the  $Q_n$ -unit on  $\Omega_a$ . If  $w$  is further a bounded solution of  $L_Q u = 0$  on  $\Omega_a$ , then the Fourier coefficients of  $w$  are bounded solutions of (6):

$$\ell_0 c_0(r, w) = \ell_n a_n(r, w) = \ell_n b_n(r, w) = 0.$$

Therefore they are represented in terms of  $Q_n$ -units:

$$\begin{cases} c_0(r, w) = c_0(a, w) f_0(r, a), \\ a_n(r, w) = a_n(a, w) f_n(r, a), \\ b_n(r, w) = b_n(a, w) f_n(r, a) \end{cases}$$

( $0 < r \leq a$ ;  $n = 1, 2, \dots$ ).

### 5. Normal derivatives of Green's functions.

We expand the inner normal derivative of the  $Q$ -Green's function into its Fourier series. For any  $\tau$  in  $[0, 2\pi)$  we denote by  $w_\tau$  a bounded solution of  $L_Q u = 0$  on  $\Omega_a$  with boundary values 1 on  $\{ae^{i\theta}; 0 < \theta < \tau\}$  and 0 on  $\{ae^{i\theta}; \tau < \theta < 2\pi\}$ . Then  $w_\tau$  is represented in an integral form:

$$w_\tau(se^{i\sigma}) = \frac{1}{2\pi} \int_0^\tau \left[ -\frac{\partial}{\partial r} G_Q^{\Omega_a}(re^{i\theta}, se^{i\sigma}) \right]_{r=a} a d\theta$$

for any  $se^{i\sigma}$  in  $\Omega_a$ . On the other hand  $w_\tau$  is represented in a Fourier series:

$$\begin{aligned} w_\tau(se^{i\sigma}) &= c_0(a, w_\tau) f_0(s, a) \\ &+ \sum_{n=1}^{\infty} \{a_n(a, w_\tau) \cos n\sigma + b_n(a, w_\tau) \sin n\sigma\} f_n(s, a). \end{aligned}$$

Since by (8) and (9) we have

$$(13) \quad \frac{f_1(s,a)}{f_0(s,a)} < 1, \quad f_n(s,a) \leq f_0(s,a) \left\{ \frac{f_1(s,a)}{f_0(s,a)} \right\}^n,$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial \tau} w_\tau(se^{i\sigma}) &= \frac{\partial}{\partial \tau} c_0(a, w_\tau) f_0(s, a) \\ &+ \sum_{n=1}^{\infty} \frac{\partial}{\partial \tau} \{a_n(a, w_\tau) \cos n\sigma + b_n(a, w_\tau) \sin n\sigma\} f_n(s, a). \end{aligned}$$

Observe that

$$\frac{\partial}{\partial \tau} c_0(a, w_\tau) = \frac{\partial}{\partial \tau} \frac{1}{2\pi} \int_0^\tau d\theta = \frac{1}{2\pi},$$

$$\frac{\partial}{\partial \tau} a_n(a, w_\tau) = \frac{\partial}{\partial \tau} \frac{1}{\pi} \int_0^\tau \cos n\theta d\theta = \frac{1}{\pi} \cos n\tau,$$

and

$$\frac{\partial}{\partial \tau} b_n(a, w_\tau) = \frac{1}{\pi} \sin n\tau.$$

Then we expand the inner normal derivative of the Q-Green's function into the following Fourier series :

$$\left[ -\frac{\partial}{\partial r} G_Q^\Omega a(re^{i\tau}, se^{i\sigma}) \right]_{r=a} = \frac{1}{a} \{f_0(s, a) + 2 \sum_{n=1}^{\infty} f_n(s, a) \cos n(\sigma - \tau)\}.$$

Estimating the right hand side of this equality by using (13) we have the following inequalities :

$$(14) \quad \left[ -\frac{\partial}{\partial r} G_Q^\Omega a(re^{i\tau}, se^{i\sigma}) \right]_{r=a} \leq \frac{1}{a} f_0(s, a) \left\{ 1 + \frac{f_1(s, a)}{f_0(s, a)} \right\} \left\{ 1 - \frac{f_1(s, a)}{f_0(s, a)} \right\}^{-1}$$

and

$$(15) \quad \left[ -\frac{\partial}{\partial r} G_Q^\Omega a(re^{i\tau}, se^{i\sigma}) \right]_{r=a} \geq \frac{1}{a} f_0(s, a) \left\{ 1 - 3 \frac{f_1(s, a)}{f_0(s, a)} \right\} \left\{ 1 - \frac{f_1(s, a)}{f_0(s, a)} \right\}^{-1}.$$

## 6. The Picard principle.

We give an example of a density on  $\Omega$  satisfying the boundary Harnack

principle, and hence the Picard principle. Let  $P$  be a general and  $Q$  a rotation free density on  $\Omega$  such that the Picard principle is valid for  $L_C$  at  $z=0$  and

$$Q(z) \leq P(z) \leq Q(z) + \frac{C}{|z|^2}$$

for a positive constant  $C$ . We take a positive integer  $k$  with  $9k^2 > C$  and consider a rotation free density  $R(z) = Q(z) + 9k^2/|z|^2$  on  $\Omega$ . First we evaluate the inner normal derivative of the  $P$ -Green's function in terms of  $Q_n$ -unit  $f_n(z,a)$  and  $R_n$ -unit  $g_n(z,a)$  on  $\Omega_a$ . Since the  $P$ -Green's function satisfies

$$G_R^{\Omega_a} \leq G_P^{\Omega_a} \leq G_Q^{\Omega_a}$$

we have

$$\begin{aligned} \left[ -\frac{\partial}{\partial r} G_R^{\Omega_a}(re^{i\tau}, se^{i\sigma}) \right]_{r=a} &\leq \left[ -\frac{\partial}{\partial r} G_P^{\Omega_a}(re^{i\tau}, se^{i\sigma}) \right]_{r=a} \\ &\leq \left[ -\frac{\partial}{\partial r} G_Q^{\Omega_a}(re^{i\tau}, se^{i\sigma}) \right]_{r=a} \end{aligned}$$

for every  $\tau$  in  $[0, 2\pi)$  and  $se^{i\sigma}$  in  $\Omega_a$ . Then by (12), (14), and (15) we obtain

$$(16) \quad \frac{\left[ -\frac{\partial}{\partial r} G_P^{\Omega_a}(re^{i\tau}, se^{i\alpha}) \right]_{r=a}}{\left[ -\frac{\partial}{\partial r} G_P^{\Omega_a}(re^{i\tau}, se^{i\beta}) \right]_{r=a}} \leq \frac{f_0(s,a)}{g_0(s,a)} \frac{1 + \frac{g_1(s,a)}{g_0(s,a)}}{1 - 3 \frac{g_1(s,a)}{g_0(s,a)}}$$

for any  $\alpha, \beta$  in  $[0, 2\pi)$  if  $g_1(s,a)/g_0(s,a) < 1/3$ .

Next we evaluate  $f_0(s,a)/g_0(s,a)$  in terms of  $g_1(s,a)/g_0(s,a)$ . From (10) it follows that

$$(17) \quad \frac{g_{4k}(s,a)}{g_0(s,a)} \geq \left\{ \frac{g_1(s,a)}{g_0(s,a)} \right\}^{(81^k - 1)/2}$$

and



$$(18) \quad \frac{f_{3k}(s,a)}{f_0(s,a)} \geq \left\{ \frac{f_1(s,a)}{f_0(s,a)} \right\}^{(27^k - 1)/2}$$

Observe that  $g_0 = f_{3k}$  and  $g_{4k} = f_{5k}$ . Then (17), (18), and

$$(19) \quad \frac{f_{5k}(s,a)}{f_{3k}(s,a)} \leq \left\{ \frac{f_1(s,a)}{f_0(s,a)} \right\}^{2k}$$

yield an evaluation

$$\frac{f_0(s,a)}{g_0(s,a)} \leq \left\{ \frac{g_0(s,a)}{g_1(s,a)} \right\}^{\alpha_k},$$

where  $\alpha_k = (81^k - 1)(27^k - 1)/8k$ .

Now we show the boundary Harnack principle (3) for  $L_p$  at  $z=0$ . We have by (17) and (19)

$$\frac{g_1(s,a)}{g_0(s,a)} \leq \left\{ \frac{f_1(s,a)}{f_0(s,a)} \right\}^{4k/(81^k - 1)}$$

Then by (11) we can take  $s_a$  in  $(0,a)$  such that  $g_1(s_a, a)/g_0(s_a, a) = 1/4$ .

Therefore by (16) we obtain

$$C(\partial\Omega_{s_a} - \{z=0\}, \Omega_a, P) \leq 5 \cdot 4^{\alpha_k}.$$

Thus (3) is valid for  $C = 5 \cdot 4^{\alpha_k}$  and  $V_U = \Omega_{s_a} \cup \{z=0\}$ , where  $a$  is a positive number with  $\Omega_a \subset U$ , so that the Picard principle is valid for  $L_p$  at  $z=0$ .

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