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# REGULARITY OF FUNCTION-KERNELS IN POTENTIAL THEORY

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## 1. Introduction

Let $G = G(x,y)$ be a strictly positive continuous function-kernel on a locally compact Hausdorff space $X$ such that every non-empty open set in $X$ is non-negligible.

When $G$ satisfies the domination principle, there exists a positive measure $\xi$ everywhere dense in $X$ satisfying

(1) $G(x,y)$ is locally $\xi$-integrable,

(2) For any f in $C_K(X)$, $V_G^{\xi} f(x) = \int G(x,y) f(y) d\xi(y)$ is in $C(X)$.

Therefore, in this case, we can consider $G$ as a continuous kernel on $x$ , that is, a positive linear form on $C_K(X)$ into $C(X)$.

We say that $G$ vanishes uniformly at infinity and write simply $G = o(1)$ unif. at $\infty$ when the following condition is satisfied:

$$\forall \varepsilon > o, \quad \forall F;\text{compact}, \quad \exists K;\text{compact} \ni$$

$$G(x,y) < \varepsilon \quad \text{on} \quad \overline{CK} \times F .$$

suppose that $G$ satisfies the complete maximum principle and that $G = o(1)$ unif. at $\infty$ . Then we can easily see that the image $V_G^{\xi}(C_K(X))$ is contained in $C_o(X)$ . Therefore, as an application of the theorem of G.Lion [9], there exists a resolvent $(V^{\xi}_{G,p})_{p>0}$ of continuous kernels associated with $V^{\xi}_{G}$ .

We denote by $V^{\tilde{\xi}}_{\check{G}}$ the continuous kernel on $X$ defined by the adjoint kernel $\check{G}$ and a measure $\tilde{\xi}$ satisfying the conditions corresponding to (1) and (2). Then, can we also associate with $V^{\tilde{\xi}}_{\check{G}}$ a resolvent $(V^{\tilde{\xi}}_{\check{G},p})_{p>0}$ of continuous kernels on $X$ ?
We stress here that the complete maximum principle is not a dual principle and that $G = o(1)$ unif. at $\infty$ does not necessarily imply $\check{G} = o(1)$ unif. at $\infty$ . Nevertheless, we can prove the following

<u>Theorem.</u> Suppose that $G$ satisfies the complete maximum principle and that $G = o(1)$ unif. at $\infty$ . Then we can associate with $V^{\xi}_{G}$ and $V^{\tilde{\xi}}_{\check{G}}$ the resolvents $(V^{\xi}_{G,p})_{p>0}$ and $(V^{\tilde{\xi}}_{\check{G},p})_{p>0}$ of continuous kernels on $X$ respectively.

In relation to the probability theory, P.A.Meyer [10], J.C.Taylor [11], R.Kondō [8] and F.Hirsch [4] developped the theory of generalized kernel, called the proper kernel, satisfying the complete maximum principle on the abstract measurable space.
For the existence of resolvent, they supposed that the kernel $V$ is regular instead of the stronger condition of Lion that $V(C_K(X))$ is contained in $C_o(X)$ .

For the proof of our theorem, we first investigate the condition for that there exists a resolvent $(V^{\xi}_{G,p})_{p>0}$ of continuous kernels associate with $V^{\xi}_{G}$ in the case that $G$ satisfies the domination principle but it does not necessarily satisfies the complete maximum principle. The duality of domination principle plays an important role in the rest of our proof.

## 2. Regularity of function-kernels

Definition 1. An l.s.c. function $u(x)$ is said to be G-supermedian when

$$\forall \mu \in E_o(G), \quad [\, G\mu(x) \leq u(x) \text{ n.e. on } S\mu \,] \Longrightarrow [\, G\mu(x) \leq u(x) \text{ on } X \,] .$$

We denote by $S(G)$ the totality of G-supermedian functions on $X$. For any $u(x)$ in $S(G)$ and for any closed set $F$ in $X$, the reduced function $R_F^G u(x)$ of $u(x)$ on $F$ is defined by

$$R_F^G u(x) = \inf \left\{ v(x) \in S(G) \; ; \; v(x) \geq u(x) \text{ n.e. on } F \right\} .$$

Definition 2. We say that $G$ is regular ( resp. weakly regular ) when we have, for any $x_o \in X$ and any compact exhaustion $(K_n)_{n=1}^{+\infty}$,

$$\lim_{n \to +\infty} R^G_{\overline{CK_n}} G\varepsilon_{x_o}(x) = o \text{ everywhere ( resp. nearly everywhere ) on } X .$$

Further we say that $G$ is strongly regular when we have for any $x_o \in X$,

$$\lim_{n \to +\infty} R^G_{\overline{CK_n}} G\varepsilon_{x_o}(x) = o \quad \text{uniformly on every compact set.}$$

Lemma 3. Let $u(x)$ be an l.s.c. function and $N = N(x,y)$ be an l.s.c. function-kernel defined by $N(x,y) = u(x)$. Then the following four statements are equivalent :

(3) $u(x)$ is G-supermedian.

(4) $G \prec N$ ( $G$ satisfies the relative domination principle with respect to $N$ ).

(5) $\check{G} \sqsubset \check{N}$ ( $\check{G}$ satisfies the transitive domination principle with respect to $\check{N}$ ).

(6) $\forall \mu \in E_o(G)$, $\forall \nu \in M_o$, $\check{G}\mu(x) \leq \check{G}\nu(x)$ n.e. on $S\mu$

$$\Longrightarrow \int u(x)\,d\mu(x) \leq \int u(x)\,d\nu(x) .$$

<u>Proof.</u> By the definition of G-supermedian function, we can easily verify the equivalence $(3) \longleftrightarrow (4)$ . On the other hand, the equivalence $(4) \longleftrightarrow (5)$ is well known. Therefore it suffices to prove that $(5) \longleftrightarrow (6)$ .

Suppose that $\check{G} \sqsubset \check{N}$ and that, for $\mu \in E_o(G)$ and $\nu \in M_o$, the inequality $\check{G}\mu(x) \leqq \check{G}\nu(x)$ holds n.e. on $S\mu$ . Then we have $\check{N}\mu(x) \sqsubset \check{N}\nu(x)$ on $X$ . Therefore

$$\int u(x)d\mu(x) = \int N\varepsilon_y(x)d\mu(x) = \int \check{N}\mu(x)d\varepsilon_y(x)$$
$$\leqq \int \check{N}\nu(x)d\varepsilon_y(x) = \int N\varepsilon_y(x)d\nu(x) = \int u(x)d\nu(x).$$

This implies (6).

Inversely we suppose (6) and that, for $\mu \in E_o(G)$ and $\nu \in M_o$, the inequality $\check{G}\mu(x) \leqq \check{G}\nu(x)$ holds n.e. on $S\mu$ . Then

$$\check{N}\mu(x) = \int N\varepsilon_x(y)d\mu(y) = \int u(y)d\mu(y) \leqq \int u(y)d\nu(y) = \int N\varepsilon_x d\nu = \check{N}\nu(x).$$

This implies $\check{G} \sqsubset \check{N}$ and hence (5) .

When $G$ satisfies the domination principle, we can express the reduced function of potential explicitly using the balayaged measure. For a measure $\mu \in M_o$ and a compact set $F$ , we denote by $B(\mu,F;G)$ the totality of balayaged measures of $\mu$ on $F$ with respect to $G$ .

<u>Lemma 4</u>. Suppose that $G$ satisfies the domination principle. Then, for any $\mu \in M_o$ and for sufficiently large $n$ , we have

$$R^G_{\overline{CK_n}} G\mu(x) = \lim_{p\to+\infty} \uparrow G\mu_{n,p}(x) ,$$

where $\mu_{n,p} \in B(\mu, \overline{CK_n} \cap K_p; G)$ .

Proof. Put $u(x) = \lim_{p\to\infty} \uparrow G\mu_{n,p}(x)$ . Then, for $\alpha \in E_o(G)$ and $\beta \in M_o$, the inequality $\check{G}\alpha(x) \leq \check{G}\beta(x)$ n.e. on $S\alpha$ implies

$$\int u d\alpha = \lim_p \int G\mu_{n,p} d\alpha = \lim_p \int \check{G}\alpha d\mu_{n,p}$$

$$\leq \lim_p \int \check{G}\beta d\mu_{n,p} = \lim_p \int G\mu_{n,p} d\beta = \int u d\beta .$$

Therefore, by virtue of Lemma 1 , $u(x)$ is G-supermedian. On the other hand, the equality $G\mu_{n,p}(x) = G\mu(x)$ n.e. on $\overline{CK_n} \cap K_p$ implies $u(x) = G\mu(x)$ n.e. on $\overline{CK_n}$ . By the minimality of the reduced function, we have

$$R^G_{\overline{CK_n}} G\mu(x) \leq u(x) = \lim_p \uparrow G\mu_{n,p}(x) .$$

Conversely, let $v(x)$ be a function in $S(G)$ such that $v(x) \geq G\mu(x)$ n.e. on $\overline{CK_n}$ . Then $G\mu_{n,p}(x) \leq v(x)$ n.e. on $\overline{CK_n} \cap K_p$ implies $u(x) \leq v(x)$ on $X$ and hence

$$\lim_p \uparrow G\mu_{n,p}(x) = u(x) \leq R^G_{\overline{CK_n}} G\mu(x)$$

because $R^G_{\overline{CK_n}} G\mu(x)$ is the infimum of such $v(x)$ .

Theorem 5. A continuous function-kernel $G$ verifying the domination principle is regular if and only if its adjoint $\check{G}$ is regular.

Proof. Let $\varepsilon_{x,n,p}$ ( resp. $\tilde{\varepsilon}_{y,n,p}$ ) be a measure in $B(\varepsilon_x, \overline{CK_n} \cap K_p; G)$ ( resp. $B(\varepsilon_y, \overline{CK_n} \cap K_p; \check{G})$ . Then we have

$G\varepsilon_{x,n,p}(y) = \check{G}\check{\varepsilon}_{y,n,p}(x)$ for sufficiently large $n$ and $p$ .

This implies

$$R^{G}_{\overline{CK_n}} G\varepsilon_x(y) = R^{\check{G}}_{\overline{CK_n}} \check{G}\varepsilon_y(x) .$$

Therefore the duality of regularity holds .

3. Existence of resolvent associated with function-kernel

A positive linear form on $C_K(X)$ into $C(X)$ is called a continuous kernel on $X$ .

For a continuous kernel $V$ on $X$ , a family $(V_p)_{p>o}$ of kernels is said to be a resolvent associated with $V$ when

(7) For any $p$ and $q > o$ ,

$$V_p - V_q = ( q-p ) V_p \cdot V_q = ( q-p ) V_q \cdot V_p ,$$

(8) $V = \lim V_p = V_o$ .

When $G$ satisfies the domination principle, we denote by $V^{\xi}_{G}$ the continuous kernel on $X$ defined by $G$ and $\xi$ satisfying the conditions (1) and (2) . Now we investigate the condition for that there exists a resolvent $(V^{\xi}_{G,p})_{p>o}$ associated with $V^{\xi}_{G}$ and constituted by continuous kernels on $X$.

As an application of the theorem of F.Hirsch [4], the author obtained the following lemma in [2] .

Lemma 6. Suppose that $G$ satisfies the complete maximum principle. If, for any $f(x)$ in $C_K(X)$ , we have

$$\lim_{n\to+\infty} R^{G}_{\overline{CK_n}} G(f\xi)(x) = o \quad \text{uniformly on every compact set,}$$

then there exists a resolvent $(V^{\xi}_{G,p})_{p>o}$ of continuous kernels on $X$ associated with $V^{\xi}_{G}$ .

We remark that $G$ satisfies the continuity principle when it satisfies the domination principle. Put

$F_o(G) = \{ \nu \in E_o(G) ; G\nu(x)$ is finite and continuous on $X \}$ .

For a measure $\alpha$ in $F_o(G)$ , we define a continuous function-kernel $K = K(x,y)$ by

$$K(x,y) = \frac{1}{G\alpha(x)} G(x,y)\ G\alpha(y) .$$

Then the following lemma is an immediate consequence of the equality:

$$R^K_{\overline{CK_n}} K\mu(x) = R^G_{\overline{CK_n}} G(G\alpha\ \mu)(x)\frac{1}{G\alpha(x)} \text{ for any } \mu \in M_o .$$

Lemma 7. The following statements are equivalent.

(9) $G$ is regular ( resp. strongly regular ).

(10) $K$ is regular ( resp. strongly regular ).

Now we can associate a resolvent with a kernel which satisfies the domination principle but does not necessarily satisfiy the complete maximum principle.

Theorem 8. Suppose that $G$ satisfies the domination principle and that $G$ is strongly regular. Then there exists a resolvent $(V^\xi_{G,p})_{p>o}$ associated with $V^\xi_G$ and constructed by continuous kernels.

Proof. Let $V^\xi_K$ be a continuous kernel on $X$ defined by

$$V^\xi_K(f\xi)(x) = \int K(x,y)f(y)d\xi(y) , \quad \forall f \in C_K(X) .$$

First we construct a resolvent $(V^\xi_{K\ p})_{p>o}$ associated with $V^\xi_K$ .

We remark that $K = K(x,y)$ satisfies the coplete maximum principle because $G$ satisfies the domination principle. Therefore, for any $f \in C_K(X)$ and any $x_o \in X$, we can fined a constant $c > o$ such that $K(f\xi)(x) < c\, G\varepsilon_{x_o}(x)$ on $X$. The kernel $G$ being strongly regular, Lemma 7 asserts that $K = K(x,y)$ is also strongly regular. Therefore there exists, for any $\varepsilon > o$ and for any compact set $F$, an integer $n_o$ veryfying

$$K\varepsilon_{x_o,n,p}(x) < \varepsilon \quad \text{on } F \text{ for every } n \text{ and } p \text{ with } p > n \geqq n_o ,$$

where $\varepsilon_{x_o,n,p}$ is a measure in $B(\varepsilon_{x_o}, \overline{CK_n} \cap K_p ; K)$. Let $(f\xi)_{n,p}$ ( resp. $\tilde{\varepsilon}_{x,n,p}$ ) be a measure in $B( f\xi , \overline{CK_n} \cap K_p ; K )$ ( resp. $B( \varepsilon_x, \overline{CK_n} \cap K_p ; \check{K} )$ ). Then for any $x \in F$ and any $n$ and $p$ with $p > n \geqq n_o$, we have

$$\begin{aligned}
K(f\xi)_{n,p}(x) &= \int \check{K}\varepsilon_x d(f\xi)_{n,p} = \int \check{K}\tilde{\varepsilon}_{x,n,p} d(f\xi)_{n,p} \\
&= \int K(f\xi)_{n,p} d\tilde{\varepsilon}_{x,n,p} = \int K(f\xi) d\tilde{\varepsilon}_{x,n,p} \\
&< c \int K\varepsilon_{x_o} d\tilde{\varepsilon}_{x,n,p} = c \int K\varepsilon_{x_o,n,p} d\tilde{\varepsilon}_{x,n,p} \\
&= c \int \check{K}\tilde{\varepsilon}_{x,n,p} d\varepsilon_{x_o,n,p} = c \int \check{K}\varepsilon_x d\tilde{\varepsilon}_{x_o,n,p} \\
&= c\, K\varepsilon_{x_o,n,p}(x) < c\varepsilon
\end{aligned}$$

and hence

$$\lim_{n\to\infty} R^K_{\overline{CK_n}} K(f\xi)(x) = o \quad \text{uniformly on every compact set.}$$

Therefore, by virtue of Lemma 6, there exists a resolvent $(V^\xi_{K,p})_{p>o}$ of continuous kernels on $X$ associated with $V^\xi_K$. Let $V^\xi_{G,p}$ be the continuous kernel on $X$ defiend by

$$V^\xi_{G,p}(f\xi)(x) = G\alpha(x)\, V^\xi_{K,p}\left( \frac{f}{G\alpha} \cdot \xi \right)(x).$$

Then we can easily see that the family $(V^{\xi}_{G,p})_{p>o}$ is the desired resolvent associated with $V^{\xi}_{G}$ and constituted by continuous kernels.

Definition 9. Let $G$ and $N$ be two continuous function-kernels. We say that $G$ vanishes uniformly and more rapidly than $N$ at infinity and write simply $G = o(N)$ unif. at $\infty$, when the following condition is fulfiled:

$$\forall F \text{ ; compact }, \ \forall \alpha \in M_o \ , \quad \exists K \text{ ; comact } \ni$$

$$G\varepsilon_x(y) < N\alpha(y) \quad \text{for} \quad x \in F \quad \text{and} \quad y \in \overline{CK}$$

Lemma 10. Suppose that $G$ satisfie the domination principle and that there exists an another continuous function-kernel $N$ satisfying

(11) $G \prec N$ ( $G$ satisfies the relative domination principle with respect to $N$ ),

(12) $G = o(N)$ unif. at $\infty$ .

Then both $G$ and $\check{G}$ are strongly regular.

Proof. First we shall prove that $G$ is strongly regular.

For any $x_o \in X$ and any compact set $F$, we can fined a measure $\alpha$ in $M_o$ such that $N\alpha(x) < \varepsilon$ on $F$. Our assumption that $G = o(N)$ unif. at $\infty$ asserts that there exists an integer $n_o$ such that $G\varepsilon_{x_o}(x) < N\alpha(x)$ on $\overline{CK_n}$ for any $n$ with $n \geqq n_o$.

Let $\varepsilon_{x_o,n,p}$ ( resp. $\tilde{\varepsilon}_{x,n,p}$ ) be a measure in $B(\varepsilon_{x_o}, \overline{CK_n} \cap K_p; G)$ ( resp. $B(\varepsilon_x, \overline{CK_n} \cap K_p; \check{G})$ ). Then, for any $n$ and $p$ with $p > n \geqq n_o$ and for any $x \in F$, we obtain

$$G\varepsilon_{x_o,n,p}(x) = \int \check{G}\varepsilon_x(y)\,d\varepsilon_{x_o,n,p}(y) = \int \check{G}\tilde{\varepsilon}_{x,n,p}(y)\,d\varepsilon_{x_o,n,p}(y)$$

$$= \int G\varepsilon_{x_o,n,p}(y)\,d\tilde{\varepsilon}_{x,n,p}(y) = \int G\varepsilon_{x_o}(y)\,d\tilde{\varepsilon}_{x,n,p}(y)$$

$$< \int N\alpha(y)\,d\tilde{\varepsilon}_{x,n,p}(y) = \int \check{N}\tilde{\varepsilon}_{x,n,p}(y)\,d\alpha(y) .$$

The inequality $\check{G}\tilde{\varepsilon}_{x,n,p}(y) \leqq \check{G}\varepsilon_x(y)$ for any $y \in X$ and the assumption (11) implies the inequality $\check{N}\varepsilon_{x,n,p}(y) \leqq \check{N}\varepsilon_x(y)$ on $X$, because (11) is equivqlent to the fact that $\check{G} \sqsubset \check{N}$. Therefore

$$G\varepsilon_{x_o,n,p}(x) < \int \check{N}\varepsilon_{x,n,p}(y)\,d\alpha(y) \leqq \int \check{N}\varepsilon_x(y)\,d\alpha(y) = N\alpha(x) < \varepsilon .$$

Hence by Lemma 4, we obtain

$$\lim_{n\to+\infty} R^G_{\overline{CK_n}} G\varepsilon_{x_o}(x) = o \quad \text{uniformly on } F .$$

Consequently $G$ is strongly regular.

We can prove that $\check{G}$ is also strongly regular. In fact, for any $x_o \in X$ and any $\varepsilon > o$, there exists a measure $\beta \in M_o$ such that $N\beta(x_o) < \varepsilon$. Our assumption that $G = o(N)$ unif. at $\infty$ asserts that we have, for any compact set $F$ and for sufficiently large $n$,

$$G\varepsilon_x(y) < N\beta(y) \quad \text{for any } x \in F \text{ and for any } y \in \overline{CK_n}.$$

Let $\tilde{\varepsilon}_{x_o,n,p}$ be a measure in $B(\varepsilon_{x_o}, \overline{CK_n} \cap K_p; \check{G})$. Then, for any $x \in F$ and sufficiently large $n$ and $p$, we have

$$\check{G}\tilde{\varepsilon}_{x_o,n,p}(x) = \int G\varepsilon_x(y)\,d\tilde{\varepsilon}_{x_o,n,p}(y) < \int N\beta(y)\,d\tilde{\varepsilon}_{x_o,n,p}(y)$$

$$= \int \check{N}\tilde{\varepsilon}_{x_o,n,p}(y)\,d\beta(y) \leqq \int \check{N}\varepsilon_{x_o}(y)\,d\beta(y) = N\beta(x_o) < \varepsilon .$$

By virtue of Lemma 4, this implies that

$$\lim_{n\to\infty} R^{\check{G}}_{\overline{CK_n}} \check{G}\varepsilon_{x_o}(x) = o \quad \text{uniformly on} \quad F$$

and hence that $\check{G}$ is strongly regular.

When $G$ satisfies the domination principle, $\check{G}$ satisfies the continuity principle. Therefore there exists a positive measure $\tilde{\xi}$ everywhere dense in $X$ satisfying

(1)' $G(x,y)$ is locally $\tilde{\xi}$-integrable,

(2)' For any $f \in C_K(X)$ , $V^{\tilde{\xi}}_{\check{G}} f(x) = \int \check{G}(x,y) f(y) d\tilde{\xi}(y)$ is in $C(X)$.

The following main theorem mentioned in section 1 is an immediate consequence of Theorem 8 and Lemma 10 .

Theorem 11. Suppose that $G$ satisfies the complete maximum principle and that $G = o(1)$ unif. at $\infty$ . Then we can associate with $V^{\xi}_G$ and $V^{\tilde{\xi}}_{\check{G}}$ the resolvents $(V^{\xi}_{G,p})_{p>o}$ and $(V^{\tilde{\xi}}_{\check{G},p})_{p>o}$ of continuous kernels on $X$ respectively.

Examples 12. We denote by $N^{\alpha}$ the riesz kernel of order $\alpha$ on $R^n$ ; $N^{\alpha} = N^{\alpha}(x,y) = |x - y|^{\alpha-n}$ ( $n \geq 3$, $o < \alpha < n$ ) .
Let $\delta$ . $\gamma$ and $\beta$ be numbers satisfying $o < \delta < \gamma < \alpha < \beta < 2$ and $\tau$ , $\nu$ and $\omega$ be measures such that $N^{\gamma}\tau(x)$ , $N^{\beta}\nu(x)$ , $N^{\delta}\omega(x)$ are finite and continuous on $X$ . Put

$$\text{(a)} \qquad G(x,y) = \frac{N^{\alpha}(x,y)}{N^{\gamma}\tau(y) + N^{\beta}\nu(y)}$$

(b) $\quad G(x,y) = \dfrac{N^{\alpha}(x,y)}{N^{\gamma}\tau(y)}$,

(c) $\quad G(x,y) = \dfrac{N^{\alpha}(x,y)}{\{N^{\gamma}\tau(y) + N^{\beta}\nu(y)\}\ N^{\beta}\nu(x)}$,

(d) $\quad G(x,y) = \dfrac{N^{\alpha}(x,y)}{N^{\gamma}\tau(x)\cdot N^{\gamma}\tau(y)}$,

(e) $\quad G(x,y) = \dfrac{N^{\alpha}(x,y)}{\{N^{\gamma}\tau(x) + N^{\beta}\nu(x)\}\cdot\{N^{\gamma}\tau(y) + N^{\beta}\nu(y)\}}$,

(f) $\quad G(x,y) = \dfrac{N^{\alpha}(x,y)}{N^{\gamma}\tau(x)\cdot N^{\delta}\omega(y)}$.

For every kernel $G$ appeared in (a) to (f), there exists a resolvent $(V^{\xi}_{G,p})_{p>o}$ ( resp. $(V^{\tilde{\xi}}_{\check{G},p})_{p>o}$ ) of continuous kernels on $X$ associated with $V^{\xi}_{G}$ ( resp. $V^{\tilde{\xi}}_{\check{G}}$ ). In (a), (b) and (c), $\check{G}$ does not satisfies the complete maximum principle. In (d) and (e), $G$ is symmetric and it does not satisfies the complete maximum principle. In (f), both $G$ and $\check{G}$ do not satisfy the complete maximum principle.

## 4. Dominated convergence property

Definition 13. We say that $G$ has the dominated convergence property ( resp. the dominated convergence property in the weak sense ) when

$$\left[\begin{array}{l} \mu_n \to \mu \quad \text{vaguely as } n \to +\infty \\ \exists \nu \in M \text{ ( resp. } \nu \in E_o(G) \text{ ) ; } G\mu_n(x) \leqq G\nu(x) \text{ on } X \text{ for all } n \end{array}\right]$$

$$\Longrightarrow \left[\, G\mu(x) = \lim_{n\to+\infty} G\mu_n(x) \quad \text{n.e. on } X \,\right].$$

In [3], we prove the following

Theorem 14. When $G$ satisfies the domination principle, the following three statements are equivalent;

(13) $G$ is weakly regular.

(14) $G$ has the dominated convergence property.

(15) $\lim_{n\to+\infty} R^{\check{G}}_{\overline{CK_n}} \check{G}\nu(x) = o$ on $X$ for every $\nu \in F_o(\check{G})$,

whre $F_o(\check{G}) = \left\{ \nu \in E_o(G);\ \check{G}\nu(x) \text{ is finite and continuous on } X \right\}$ .

The author does not know whether the duality of the weak regularity or that of the dominated convergence property hols or not. But we can prove the duality when we limit our argument to the property in the weak sense.

Theorem 15. If $G$ satisfies the domination principle, the following four statements are equivqlent :

(16) $G$ has the dominated convergence property in the weak sense.

(17) $\check{G}$ has the dominated convergence property in the weak sense.

(18) $\lim_{n\to+\infty} R^{G}_{\overline{CK_n}} G\nu(x) = o$ n.e. on $X$ for every $\nu \in F_o(G)$ .

(19) $\lim_{n\to+\infty} R^{\check{G}}_{\overline{CK_n}} \check{G}\nu(x) = o$ n.e. on $X$ for every $\nu \in F_o(\check{G})$ .

Proof. It suffices to obtain the implications $(16)\longrightarrow(19)$ $(19)\longrightarrow(17)$ .

$(16)\longrightarrow(19)$ . Let $\nu \in F_o(\check{G})$ and $\tau \in E_o(G)$ . By (16), we can construct a balayaged measure $\tau_n$ of $\tau$ on $\overline{CK_n}$.

For anf $\tilde{\nu}_{n,p}$ $B(\nu, \overline{CK_n} \cap K_p; \check{G})$ , we have

$$\int \check{G}\tilde{\nu}_{n,p} d\tau = \int G\tau d\tilde{\nu}_{n,p} = \int G\tau_n d\tilde{\nu}_{n,p} = \int \check{G}\nu d\tau_n = \int G\tau_n d\nu .$$

This implies (19), because $(\tau_n)$ converges vaguely to 0 and $G\tau_n(x)$ is dominated uniformly by $G\tau(x)$. Thus we have (16)$\longrightarrow$(19).

(19)$\longrightarrow$(17). Suppose that $\mu_k \longrightarrow \mu$ vaguely as $k \longrightarrow +\infty$ and that there exists a measure $\nu \in E_0(G)$ such that $\check{G}\mu_k(x) \leqq \check{G}\nu(x)$ for any k. Then, for any $\tau \in F_0(G)$, we have

$$\int \check{G}\mu_k d\tau = \int G\tau d\mu_k \leqq \int_{K_n} G\tau d\mu_k + \int_{\overline{CK_n}} G\tau d\mu_k .$$

We remark

$$\int_{\overline{CK_n}} G\tau d\mu_k = \lim_{p\to+\infty} \int_{\overline{CK_n}\cap K_p} G\tau d\mu_k ,$$

and $$\int_{\overline{CK_n}\cap K_p} G\tau d\mu_k = \int G\tau_{n,p} d\mu_k = \int \check{G}\mu_k d\tau_{n,p} \leqq \int \check{G}\nu d\tau_{n,p}$$

$$\leqq \int R^{\check{G}}_{\overline{CK_n}} \check{G}\nu(x) d\tau_{n,p} \leqq \int R^{\check{G}}_{\overline{CK_n}} \check{G}\nu d\tau ,$$

where the last inequality is derived from Lemma 3. Therefore, by (19), we have, for sufficiently large n,

$$\int_{\overline{CK_n}} G\tau d\mu_k \leqq \int R^{\check{G}}_{\overline{CK_n}} \check{G}\nu d\tau < \varepsilon .$$

Hence

$$\lim_{k\to+\infty} \int \check{G}\mu_k d\tau \leqq \lim_{k\to+\infty} \int_{K_n} G\tau d\mu_k + \varepsilon \leqq \int G\tau d\mu + \varepsilon .$$

This implies that $\check{G}\mu(x) \geqq \underline{\lim}\, \check{G}\mu_k(x)$ n.e. on X. The inverse inequality is obvious. Therefore we have (19)$\longrightarrow$(17).

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