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On the minimal thinness in a Lipschitz domain

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§1. Introduction

Let H be an open half space in R^n, $n \geqq 3$, and ξ be a point on the boundary of H. Jackson [5; Théorème 1] proved

Theorem A. Let $E \subset H$. (i) If E is thin at ξ in the ordinary sense, then E is minimally thin at ξ.

(ii) If E is contained in a nontangential cone with vertex at ξ, then the converse of (i) holds.

The main aim of this paper is to extend the above theorem to a bounded Lipschitz domain D in R^n, $n \geqq 3$. A bounded domain D is said to be Lipschitz if ∂D is covered by finitely many open circular cylinders whose bases have positive distance from ∂D and for each cylinder Ψ there are a function $\phi: R^{n-1} \to R$ satisfying a Lipschitz condition and a congruent transformation T such that

$$(1.1)\qquad \begin{aligned} T(\Psi \cap D) &= \{(x', x_n);\ x_n > \phi(x')\} \cap T(\Psi), \\ T(\Psi \cap \partial D) &= \{(x', x_n);\ x_n = \phi(x')\} \cap T(\Psi), \end{aligned}$$

where $x' = (x_1, \ldots, x_{n-1}, 0)$.

It is known that the Martin compactification of D is homeomorphic to the Euclidean closure $\overline{D}$ and that every point $\xi \in$

∂D is a minimal boundary point (see Hunt and Wheeden [4]). Let $G(x, y)$ be the Green function for D. Letting z_0 be fixed, we put $g(x) = G(x, z_0)$. The Martin kernel $K(x, y)$ for D with the reference point z_0 is defined by

$$K(x, y) = \begin{cases} G(x, y)/g(y) & \text{if } (x, y) \in D \times (D \setminus \{z_0\}) \\ \lim_{Y \to y} G(x, Y)/g(Y) & \text{if } (x, y) \in D \times \partial D. \end{cases}$$

The minimal thinness is defined by using the Martin kernel as follows: Let $\xi \in \partial D$. A subset E of D is said to be minimally thin at ξ if $\hat{R}^E_{K(\cdot, \xi)} < K(\cdot, \xi)$. Here $\hat{R}^E_{K(\cdot, \xi)}$ stands for the regularized reduced function of $K(\cdot, \xi)$ relative to E (see Brelot [2; p.122]). We shall show

Theorem 1.1. Let D be a bounded Lipschitz domain in R^n, $n \geqq 3$. Then (i) and (ii) of Theorem A are valid for each point ξ on the boundary of D.

Although the minimal thinness is originally defined by the Martin kernel, we shall in the proof of the theorem, use another characterization of the minimal thinness, which was shown in Naïm [10; Chapitre II]. She introduced a kernel Θ by

$$\Theta(x, y) = \begin{cases} G(x, y)/[g(x)g(y)] & \text{if } (x, y) \in (D \setminus \{z_0\}) \times (D \setminus \{z_0\}) \\ \liminf_{X \to x,\ Y \to y} G(X, Y)/[g(X)g(Y)] & \\ & \text{if } (x, y) \in (\bar{D} \times \bar{D}) \setminus (D \times D), \end{cases}$$

and proved that E is minimally thin at ξ if and only if there exists a Θ-potential $\Theta(x, \mu) = \int \Theta(x, y)d\mu(y) \not\equiv \infty$ such that

$$(1.2) \qquad \lim_{x\to\xi,\ x\in E} \Theta(x, \mu)/\Theta(x, \xi) = \infty$$

([10; Théorème 8]). We shall give several estimates of Θ in §3, which play important roles in the proof of Theorem 1.1. For this purpose we shall use the boundary Harnack principle (see Wu [13; Theorem 1]).

It is well known that the ordinary thinness is characterized by Wiener criterion. A Wiener's type criterion will be a useful tool in the final stage of the proof of Theorem 1.1 in §4. By the aid of the estimates of Θ in §3, we shall find that the minimal thinness is characterized by a Wiener's type criterion associated with Naïm's Θ-kernel, and that a relationship between the Newtonian capacity and $C_{\Theta,1}$-capacity (see §2) leads to Theorem 1.1.

In §2, however, we shall deal with not only the Wiener's type criterion for the minimal thinness but also that for the thinness with respect to L^p-potentials, which is a slight generalization of the (k, p)-thinness in Meyers [7]. This is partly because the arguments can be carried out similarly, but chiefly because the Wiener's type criterion for L^p-potentials and the estimates of Θ in §3 will enable us to describe the behavior of Green potentials with L^p-densities. Let $G(x, f) = \int_D G(x, y)f(y)dy$ for a nonnegative measurable function f on D. We shall prove

Theorem 1.2 (cf. [12; Corollary 6.3]). Let $G(x, f) \not\equiv \infty$ and

$$\int_D f(x)^p \delta(x)^{2p-1} dx < \infty, \tag{1.3}$$

where $\delta(x)$ stands for the distance from x to ∂D. Then there is a subset F of ∂D, of vanishing $(n - 1)$-dimensional Hausdorff measure, such that for each $\xi \in \partial D \setminus F$, there is a set $\tilde{E}_\xi \subset \{x; |x| = 1\}$ satisfying $B_{2,p}(\tilde{E}_\xi) = 0$ and

$$\lim_{r\to 0} G(\xi + r\alpha, f) = 0$$

for all $\alpha \in N(\xi) \setminus \tilde{E}_\xi$. Here $B_{2,p}$ stands for the Bessel capacity of index $(2, p)$ (see [6; Definition 16]) and $N(\xi)$ denotes the totality of all nontangential unit vectors at ξ. If $2p > n$, then $\tilde{E}_\xi$ is empty; moreover, $G(x, f)$ has nontangential limit zero at ξ.

§2. Preliminaries

We shall use the following notation: $B(x, r)$ (resp. $C(x, r)$) is the open (resp. closed) ball in R^n whose center is at x and whose radius is r. Letting s, $0 < s < 1$, be fixed, we set $I_j(\xi) = C(\xi, s^j) \setminus C(\xi, s^{j+1})$ and $I_j^*(\xi) = I_{j-1}(\xi) \cup I_j(\xi) \cup I_{j+1}(\xi)$ for a point $\xi \in R^n$ and an integer j. If E is a subset of R^n, then $E(\xi, j)$ denotes the part of E in $I_j(\xi)$, i.e., $E(\xi, j) = E \cap I_j(\xi)$. We shall abbreviate $I_j(0)$, $I_j^*(0)$ and $E(0, j)$ to I_j, I_j^* and $E(j)$, respectively. The characteristic function of E is denoted by $\chi(E)$.

If μ is a measure on a Borel set E, f is a (Borel) measurable function on E and $k(x, y)$ is a measurable function on $R^n \times R^n$, then $k(x, \mu)$ and $k(x, f)$ stand for $\int_E k(x, y)d\mu(y)$ and $\int_E k(x, y)f(y)dy$ respectively, provided they are well defined. Let $\|f\|_p$ be the usual L^p-norm of f and $\|\mu\|_1$ be the total variation of μ. If U is a Borel set in R^n, then $L^p(U) = \{f; \|f\|_p < \infty,$ $f = 0$ on $R^n \setminus U\}$, $L^p_+(U) = \{f \in L^p(U); f \geqq 0$ a.e. on $U\}$, $M(U) = \{\mu; \mu(R^n \setminus U) = 0\}$ and $M_+(U) = \{\mu \in M(U); \mu \geqq 0\}$.

For simplicity we let A stand for a positive constant independent of the variables involved in inequalities where A appears, possibly changing from one occurrence to the next. The symbol $\sim$ between two positive functions means that their ratio is bounded above and below by positive constants independent of the variables involved.

Let $0 < r \leqq \infty$ and ϕ be a positive continuous function on $(0, r)$. We say that ϕ is moderate on $(0, r)$, if there is a constant $A \geqq 1$ such that

$$A^{-1} \leqq \phi(t)/\phi(\sigma t) \leqq A \quad \text{for all } t \in (0, r) \text{ and all } \sigma \in [s, 1],$$

with s appearing in the definition of $I_j(\xi)$. We observe that s can be replaced by any constant $s' \in (0, 1)$ in the above definition. We just say that ϕ is moderate if ϕ is moderate on $(0, r)$ for some $r > 0$. Let $\rho(\phi, r) = \limsup_{\sigma \to 0} \sup_{0<t<r} \phi(t)/\phi(\sigma t)$. We see that if $\rho(\phi, r) < 1$, then $\lim_{t \to 0} \phi(t) = \infty$. We simply write $\rho(\phi) < 1$ if $\rho(\phi, r) < 1$ for some $r > 0$.

Unless otherwise specified, the letters ξ, j, k, p and ϕ will stand for a point in R^n, an integer, a nonnegative measurable function on $R^n \times R^n$, a number which is not smaller than 1 and a moderate function, respectively.

Following Meyers [6; Definition 6], we introduce an L^p-capacity. Let U be a Borel set in R^n. We put

$$C_{k,p}(E; U) = \inf\{\|f\|_p^p;\ k(x, f) \geqq 1 \text{ on } E,\ f \in L_+^p(U)\}, \quad \text{if } p > 1,$$

$$C_{k,1}(E; U) = \inf\{\|\mu\|_1;\ k(x, \mu) \geqq 1 \text{ on } E,\ \mu \in M_+(U)\}.$$

We abbreviate $C_{k,p}(E; R^n)$ to $C_{k,p}(E)$. If $U \subset U' \subset R^n$, then $C_{k,p}(E; U) \geqq C_{k,p}(E; U') \geqq C_{k,p}(E)$ for all $E \in R^n$.

Meyers [7; Definition 3.1] introduced the notion of (k, p)-thinness of k of the form $k(x, y) = \kappa(|x - y|)$, where κ is a positive nonincreasing function on $R^+ = \{x; x > 0\}$. We need to generalize this notion to nonnegative measurable function k on $R^n \times R^n$. Let $q > 0$.

Definition 2.1. A set E is said to be (k, p, q, ϕ)-thin (resp. (k, p, ϕ)-semithin) at ξ if

$$\Sigma_{j=j_0}^{\infty} [\phi(s^j)^p C_{k,p}(E(\xi, j))]^{q/p} < \infty$$

$$(\text{resp. } \lim_{j\to\infty} \phi(s^j)^p C_{k,p}(E(\xi, j)) = 0),$$

where j_0 is some positive integer such that ϕ is moderate on $(0, s^{j_0})$

This definition is independent of the choice of s, $0 < s < 1$. We shall sometimes suppose that s is small enough. If E is (k, p, q, ϕ)-thin at ξ, then obviously E is (k, p, ϕ)-semithin at ξ.

Remark 2.1. If $k(x, y) = \kappa(|x - y|)$ and $\hat{\phi}_p(t) = \phi(t)t^{n(p-1)/p}$, then the $(k, p, p/(p-1), \hat{\phi}_p)$-thinness is identical with the (k, p)-thinness in the notation of Meyers [7; Definition 3.1]. If $U_\alpha(x, y) = |x - y|^{\alpha-n}$, $1 \leqq p < n/\alpha$, then the $(U_\alpha, p, t^{\alpha-n/p})$-semitninness is identical with the (α, p)-semithinness in the notation of Mizuta [9; §1]. The ordinary thinness (resp. semithinness) is identical with the $(U_2, 1, 1, t^{2-n})$-thinness (resp. $(U_2, 1, t^{2-n})$-semithinness) in case $n \geqq 3$.

The next lemma is a generalization of [7; Proposition 3.1]. We omit the proof.

Lemma 2.1. If $\{E_i\}_{i=1}^{\infty}$ is a sequence of sets which are (k, p, q, ϕ)-thin (resp. (k, p, ϕ)-semithin) at ξ, then there exists a sequence $\{j(i)\}_{i=1}^{\infty}$ such that $E = \cup_{i=1}^{\infty} \cup_{j=j(i)}^{\infty} E_i(\xi, j)$ is (k, p, q, ϕ)-thin (resp. (k, p, ϕ)-semithin) at ξ.

Hereafter we shall deal with k satisfying for some s, $0 < s < 1$, and hence for all s, $0 < s < 1$,

$$(2.1) \qquad k(x, y) \leqq A \begin{cases} \phi(|y - \xi|) & \text{if } |x - \xi| \leqq s|y - \xi| \\ \\ \phi(|x - \xi|) & \text{if } |y - \xi| \leqq s|x - \xi|, \end{cases}$$

where A is independent of x and y, and ϕ is a moderate function on $(0, \infty)$. We have

Proposition 2.1. Let $\hat{\phi}_p(t) = \phi(t)t^{n(p-1)/p}$ as in Remark 2.1. Suppose that k satisfy (2.1) and $\rho(\hat{\phi}_p, \infty) < 1$. Then E is $(k, p, q, \hat{\phi}_p)$-thin (resp. $(k, p, \hat{\phi}_p)$-semithin) at ξ if and only i

$$\Sigma_{j=j_0}^{\infty} [\hat{\phi}_p(s^j)^p C_{k,p}(E(\xi, j); I_j^*(\xi))]^{q/p}$$

$$(\text{resp. } \lim_{j\to\infty} \hat{\phi}_p(s^j)^p C_{k,p}(E(\xi, j); I_j^*(\xi)) = 0),$$

where j_0 is chosen as in Definition 2.1.

We consider the behavior of a potential k(x, f) under a certain condition on f. Noting Definition 2.1, we introduce

Definition 2.2. Let $M(\phi, p, q, \xi)$ (resp. $N(\phi, p, \xi)$) be the totality of all locally integrable functions f $(p > 1)$ and measures μ $(p = 1)$ such that

$$\Sigma_{j=j_0}^{\infty} [\phi(s^j)^p \int_{B(\xi, s^j)} |f(y)|^p dy]^{q/p} < \infty$$

$$(\text{resp. } \lim_{j\to\infty} \phi(s^j)^p \int_{B(\xi, s^j)} |f(y)|^p dy = 0) \qquad \text{for } p > 1,$$

$$\Sigma_{j=j_0}^{\infty} [\phi(s^j)|\mu|(B(\xi, s^j))]^q < \infty$$

$$(\text{resp. } \lim_{j\to\infty} \phi(s^j)|\mu|(B(\xi, s^j)) = 0) \qquad \text{for } p = 1,$$

where j_0 is chosen as in Definition 2.1 and $|\mu|$ is the total variation of μ.

This definition is independent of the choice of s. It is clear that $M(\phi, p, q, \xi) \subset N(\phi, p, \xi)$. The main theorem of this section is

Theorem 2.1. Let k satisfy (2.1) and ψ be a nonincreasing moderate function satisfying $\rho(\phi/\psi) < 1$.

(i) If $f \in M(\hat{\psi}_p, p, q, \xi)$ and

$$\text{(2.2)} \qquad \begin{cases} \int_{|y-\xi| \geqq r} \psi(|y - \xi|)|f(y)|dy < \infty & \text{in case } p > 1, \\ \int_{|y-\xi| \geqq r} \psi(|y - \xi|)d|f|(y) < \infty & \text{in case } p = 1, \end{cases}$$

for some $r > 0$, and hence for all $r > 0$, then there exists a set E $(k, p, q, \hat{\phi}_p)$-thin at ξ such that

$$\text{(2.3)} \qquad \lim_{x \to \xi,\ x \notin E} [\psi(|x - \xi|)/\phi(|x - \xi|)]k(x, f) = 0.$$

(ii) If $f \in N(\hat{\psi}_p, p, \xi)$ and it satisfies (2.2), then there exists a set E $(k, p, \hat{\phi}_p)$-semithin at ξ such that (2.3) holds.

Here, if E includes $B(\xi, r) \setminus \{\xi\}$ for some $r > 0$, then the left hand side of (2.3) is understood to be equal to zero. The theorem follows from Lemma 2.1 and

Lemma 2.2. Let k, ϕ, and ψ be as above and $c > 0$.

(i) If $f \in M(\hat{\psi}_p, p, q, \xi)$ and it satisfies (2.2), then there exists a set E $(k, p, q, \hat{\phi}_p)$-thin at ξ such that

$$\text{(2.4)} \qquad \sup_{B(\xi, 1) \setminus E} [\psi(|x - \xi|)/\phi(|x - \xi|)]k(x, |f|) < c.$$

(ii) If $f \in N(\hat{\psi}_p, p, \xi)$ and it satisfies (2.2), then there exists a set E $(k, p, \hat{\phi}_p)$-semithin at ξ such that (2.4) holds.

Proof. Without loss of generality, we may assume that ξ =

0 and $f \geqq 0$. Suppose that $f \in N(\hat{\psi}_p, p, \xi)$ and satisfies (2.2). We claim that

$$(2.5) \qquad \lim_{j\to\infty} [\psi(s^j)/\phi(s^j)] \sup_{x\in I_j} k(x, f-f_j) = 0,$$

where $f_j = f\chi(I_j^*)$. We shall prove (2.5) only in case $p > 1$, because the case $p = 1$ can be proved similarly. Since $\rho(\phi/\psi) < 1$ it follows that $\lim_{t\to 0} \psi(t)/\phi(t) = 0$. On account of (2.1) and (2.2), we have for all $r > 0$

$$\lim_{x\to 0} [\psi(|x|)/\phi(|x|)] k(x, f|_{B(0, r)^c}) = 0.$$

Hölder's inequality leads to

$$\lim_{j\to\infty} \psi(s^j) \int_{B(0, s^j)} f(y)dy = 0.$$

We have by (2.1)

$$\limsup_{j\to\infty} [\psi(s^j)/\phi(s^j)] \sup_{x\in I_j} k(x, f\chi(\cup_{i=2}^{\infty} I_{j+i}))$$
$$\leqq A \limsup_{j\to\infty} \psi(s^j) \int_{B(0, s^j)} f(y)dy = 0.$$

Take $s > 0$ and $r > 0$ so that $\eta = \sup_{0<t<r} [\phi(t)\psi(st)]/[\phi(st)\psi(t)] < 1$. If $x \in I_j$, $y \in I_{j-i}$, $2 \leqq i \leqq j - j_0$ and $s^{j_0} < r$, then (2.1) yields

$$[\psi(s^j)/\phi(s^j)] k(x, y) \leqq A[\psi(s^j)/\phi(s^j)]\phi(s^{j-i})$$
$$\leqq A\eta^i \psi(s^{j-i}).$$

Since $0 < \eta < 1$,

$$\limsup_{j\to\infty} [\psi(s^j)/\phi(s^j)] \sup_{x\in I_j} k(x, f\chi(\cup_{i=2}^{j-j_0} I_{j-i}))$$
$$\leqq A \limsup_{j\to\infty} \Sigma_{i=2}^{j-j_0} \eta^i \psi(s^{j-i}) \int_{I_{j-i}} f(y)dy$$

$$\leqq A \limsup_{j\to\infty} \Sigma_{i=0}^{j} \eta^{j-i}\psi(s^i)\int_{B(0,\ s^i)} f(y)dy = 0.$$

Thus (2.5) follows.

Since ϕ and ψ are moderate, there is a constant A independent of j such that $\psi(s^j)/\phi(s^j) \geqq A \sup_{x\in I_j} [\psi(|x|)/\phi(|x|)]$. In view of (2.5), we can find a positive integer N such that if $j \geqq N$, then

$$E_j = \{x \in I_j;\ [\psi(|x|)/\phi(|x|)]k(x,\ f) \geqq c\}$$
$$\subset \{x \in I_j;\ [\psi(s^j)/\phi(s^j)]k(x,\ f) \geqq Ac\}$$
$$\subset \{x \in I_j;\ [\psi(s^j)/\phi(s^j)]k(x,\ f_j) \geqq Ac/2\}.$$

Therefore

$$C_{k,p}(E_j) \leqq (Ac/2)^{-p}[\psi(s^j)/\phi(s^j)]^p\int_{I_j^*} f(y)^p dy$$

$$\leqq (Ac/2)^{-p}[\hat{\psi}_p(s^j)/\hat{\phi}_p(s^j)]^p\int_{B(0,\ s^{j-1})} f(y)^p dy.$$

Let $E = \cup_{j=0}^{\infty} E_j$. If f satisfies the hypothesis of (i) (resp. (ii)), then E is $(k,\ p,\ q,\ \hat{\phi}_p)$-thin (resp. $(k,\ p,\ \hat{\phi}_p)$-semithin) at 0 and (2.4) holds. The proof is complete.

The size of the exceptional set in Theorem 2.1 is best possible in the following sense. The proof will be given elsewhere.

Theorem 2.2. Let k satisfy (2.1) and $\rho(\hat{\phi}_p,\ \infty) < 1$. If E is $(k,\ p,\ q,\ \hat{\phi}_p)$-thin (resp. $(k,\ p,\ \hat{\phi}_p)$-semithin) at ξ and ξ is not an isolated point of $E \cup \{\xi\}$, then there exists $f \in M(\hat{\psi}_p,\ p,\ q,\ \xi)$ (resp. $N(\hat{\psi}_p,\ p,\ \xi)$) such that $f \geqq 0$, supp f is compact and

$$\lim_{x\to\infty,\ x\in E} [\psi(|x - \xi|)/\phi(|x - \xi|)]k(x,\ f) = \infty.$$

§3. Estimates of Naïm's Θ-kernel

Let D be a bounded Lipschitz domain as in §1 and let S be a portion of ∂D which is expressed by (1.1). For simplicity we assume that T is the identity. For $\xi \in S$ and $r > 0$, we put

$$\Psi(\xi, r) = \{(x', x_n) \in R^{n-1} \times R; |x' - \xi'| < r, |x_n - \xi_n| < mr\},$$

where $m > 1$ is a positive constant large enough as in Hunt and Wheeden [4; p. 510], depending only on D. We can choose $r_0 > 0$ and $\xi_0 \in S$ so that $\Psi(\xi, 4mr_0) \subset \Psi$ for all $\xi \in S_0 = S \cap B(\xi_0, r_0)$ and $B(z_0, 2r_0) \subset D \setminus \Psi$, where Ψ is the cylinder in (1.1) and z_0 is the point in the Definition $K(x, y)$.

The set $\Gamma(\xi, \alpha, a) = \{x \in R^n; (x - \xi, \alpha) > a|x - \xi|\}$ with a unit vector α and a constant a, $-1 < a < 1$, is called a cone with vertex at ξ and axis along α, where $(\cdot, \cdot)$ denotes the inner product. If there are $r > 0$ and a', $-1 < a' < a$, such that $\Gamma(\xi, \alpha, a') \cap B(\xi, r) \subset D$, then $\Gamma(\xi, \alpha, a)$ is said to be a nontangential cone of D with vertex at ξ and α is said to be a nontangential unit vector at ξ. We observe that $e = (0, \ldots, 0, 1)$ is a nontangential unit vector at $\xi \in S$.

We put $\tau_\xi(r) = g(\xi + re)$ with $g(\cdot) = G(\cdot, z_0)$ and $\phi_\xi(r) = r^{2-n}\tau_\xi(r)^{-2}$ for $r > 0$ such that $\xi + re \in \Psi$. For convenience, we put $\tau_\xi(r) = 0$ and $\phi_\xi(r) = +\infty$. If α is a different nontangential unit vector at ξ, then by the Harnack principle we have $g(\xi + r\alpha) \sim \tau_\xi(r)$ for $r > 0$. The Harnack principle also implies that ϕ_ξ is moderate. Throughout this section we let $\xi \in S_0$. We shall show that Θ satisfies (2.1) with $\phi = \phi_\xi$ at ξ.

A useful tool is the boundary Harnack principle, which was independently proved by Dahlberg [3], Wu [13] and Ancona [1]. We need a slight stronger form of the boundary Harnack principle, which can be immediately deduced from [13; Lemma 7]. This is

Lemma 3.1. Let $r < r_0$. Suppose that u and v are positive harmonic functions on $\tilde{D} = D \cap \Psi(\xi, 4mr) \setminus \Psi(\xi, (4m)^{-1}r)$ which vanish on $\partial\tilde{D} \cap \partial D$. If $u(\xi + mre) \leqq v(\xi + mre)$, then

$$u(x) \leqq Av(x) \quad \text{for } x \in D \cap \Psi(\xi, 2mr) \setminus \Psi(\xi, (2m)^{-1}r),$$

where A is a positive constant independent of u, v, ξ and r.

By the aid of Lemma 3.1, we have

Lemma 3.2. Let x and $y \in B(\xi, r_0) \cap \overline{D}$. If $4m^2|y - \xi| \leqq |x - \xi|$, then with constants of comparison independent of x and y,

$$\Theta(x, y) \sim \phi_\xi(|x - \xi|).$$

In particular $\Theta(x, \xi) \sim \phi_\xi(|x - \xi|)$.

Proof. In view of the definition of Θ, we may assume that x and y are points in D. Let $|x - \xi| = mr$ and $x^* = \xi + mre$. Since $m > 1$, we observe that $x \in \Psi(\xi, 2mr) \setminus \Psi(\xi, (2m)^{-1}r)$ and that $y \in \Psi(\xi, (4m)^{-1}r)$. Since $G(\cdot, y)$ is positive and harmonic in $D \setminus \Psi(\xi, (4m)^{-1}r)$ and vanishes on ∂D, it follows from [4; (2.4)] that

$$g(y) \leqq AG(x^*, y)\omega^{z_0}(\Psi(\xi, r) \cap \partial D),$$

where $\omega^{z_0}(\cdot)$ is the harmonic measure at z_0 with respect to D. On account of [3; Lemma 1], we have

$$(3.1) \qquad g(y) \leqq AG(x^*, y)r^{n-2}g(x^*).$$

Hence,

$$G(x^*, y) \geqq A\phi_\xi(mr)g(y)g(x^*).$$

On the other hand, letting $B = B(x^*, 2^{-1}mr)$, we have

$$G(x^*, z) \leqq |x^* - z|^{2-n} \leqq A\phi_\xi(mr)g(x^*)^2 \qquad \text{for } z \in \partial B.$$

By the Harnack principle, $g(z) \sim g(x^*)$ for $z \in \partial B$, so that

$$G(x^*, z) \leqq A\phi_\xi(mr)g(x^*)g(z) \qquad \text{for } z \in \partial B.$$

Since $B \cap \Psi(\xi, (4m)^{-1}r) = \emptyset$, the maximum principle leads to

$$G(x^*, y) \leqq A\phi_\xi(mr)g(x^*)g(y).$$

Let $u(z) = \phi_\xi(mr)g(z)g(y)$ and $v(z) = G(z, y)$. Then both u and v are positive and harmonic in $\tilde{D}$, vanish on $\partial\tilde{D} \cap \partial D$ and $u(x^*) \sim v(x^*)$. On account of Lemma 3.1, we have

$$u(z) \sim v(z) \qquad \text{for } z \in D \cap \Psi(\xi, 2mr) \setminus \Psi(\xi, (2m)^{-1}r),$$

so that

$$\Theta(x, y) \sim \phi_\xi(mr) = \phi_\xi(|x - \xi|).$$

The lemma follows.

Now we list up several further estimates of Θ. For lack of space we omit the proofs, which will be given in a seperate paper.

Lemma 3.3. If $x \in B(\xi, r_0/(4m^2)) \cap \overline{D}$ and $y \in \overline{D} \setminus B(\xi, r_0)$, then with constants of comparison independent of x and y,

$$\Theta(x, y) \sim |y - z_0|^{n-2}.$$

Lemma 3.4. Let $\Gamma(\xi, e, a)$ be a nontangential cone of D with vertex at ξ and $a < a' < 1$. If $x \in \Gamma(\xi, e, a') \cap B(\xi, r_0)$, $y \in \overline{D} \cap B(\xi, r_0) \setminus \Gamma(\xi, e, a)$, then

$$\Theta(x, y) \leqq A\phi_\xi(|x - \xi|),$$

where A is independent of x and y. If furthermore, $|x - \xi| \sim |y - \xi|$, then

$$\Theta(x, y) \leqq A\phi_\xi(|x - y|).$$

Lemma 3.5. Let Γ be a nontangential cone of D with vertex at ξ and let x and y be points in $\Gamma \cap B(\xi, r_0)$. If $|x - \xi| \sim |y - \xi|$, then

$$\Theta(x, y) \sim \tau_\xi(|x - \xi|)^{-2}|x - y|^{2-n}.$$

Lemma 3.6. Let x and y be points in $\overline{D} \cap B(\xi, r_0)$. If $|x - \xi| \sim |y - \xi|$, then

$$\Theta(x, y) \geqq A\tau_\xi(|x - \xi|)^{-2}|x - y|^{2-n}.$$

We conclude this section by showing an estimate of g. It is well known that if ∂D is sufficiently smooth, then $g(x)$ is comparable to dist $(x, \partial D)$. For a Lipschitz domain this does not hold. Nevertheless we have

Lemma 3.7. There are positive constants δ and A depending only on D such that

$$(3.2) \qquad g(\xi + \sigma re)/g(\xi + re) \leqq A\sigma^{\delta}$$

for all $\xi \in S_0$, all r, $0 < r \leqq r_0$, and σ, $0 < \sigma < 1$. In particular, if $\beta < n - 2 + 2\delta$, then $\rho(\phi_\xi(r)r^{\beta}) < 1$ for all $\xi \in S_0$.

Proof. We can find a positive constant a such that $\Gamma(\xi, -e, a/2) \cap D = \emptyset$ for every $\xi \in S_0$. Let h be the harmonic function on $R^n \setminus \Gamma(0, -e, a)$ such that $h(e) = 1$ and h vanishes on $\partial\Gamma(0, -e, a)$. Then $h(x)$ is of the form $|x|^{\delta}h(x/|x|)$ for some positive constant δ depnding only on a and the dimension n. From the Harnack principle it follows that

$$A|x|^{\delta} \leqq h(x) \leqq |x|^{\delta} \qquad \text{for } x \in R^n \setminus \Gamma(0, -e, a/2).$$

Let v be the harmonic measure of $\partial B(\xi, 2r) \cap D$ in $B(\xi, 2r) \cap D$. The maximum principle and the boundary Harnack principle leads to

$$g(x) \leqq Ag(\xi + re)v(x) \leqq Ag(\xi + re)r^{-\delta}h(x - \xi)$$

for $x \in B(\xi, r) \cap D$. Hence if $x = \xi + \sigma re$ and $0 < \sigma < 1$, then (3.2) follows. Since $\phi_\xi(r) = r^{2-n}\tau_\xi(r)^{-2} = r^{2-n}g(\xi + re)^{-2}$, we have the second assertion immediately.

§4. Proofs of Theorems 1.1 and 1.2

It is sufficient to prove Theorem 1.1 only for points ξ in S_0. We assume that the constant s appearing in the definition $I_j(\xi)$ is smaller than $(4m^2)^{-1}$. It is convenient, for the sake of application of the results in §2, to extend the Naïm's Θ-kernel defined in §1 to a function on $R^n \times R^n$ which is equal to Θ on $\overline{D} \times \overline{D}$ and vanishes on the complement of $\overline{D} \times \overline{D}$. We also write Θ for the extended function. We extend ϕ_ξ, which was originally defined for small positive r, to a moderate function on $R^+ = \{r;\ r > 0\}$ so as to satisfy $\rho(\phi_\xi, \infty) < 1$. This is possible on account of Lemma 3.7. In particular, we see that $\lim_{t\to 0} \phi_\xi(t) = +\infty$. Since $\Theta(x, y)$ is symmetric, it follows from Lemma 3.2 that Θ satisfies (2.1) with $\phi = \phi_\xi$ at $\xi \in S_0$. We infer from Lemma 3.3

Lemma 4.1. Let μ be a measure on $\overline{D}$. Then $\Theta(x, \mu) \not\equiv \infty$. Furthermore, if the support of μ does not contain ξ, then $\Theta(x, \mu)$ is bounded when x tends to ξ.

Proof of Theorem 1.1. First we show that E is minimally thin at ξ if and only if E is $(\Theta, 1, 1, \phi_\xi)$-thin at ξ.

If E is minimally thin at ξ, then there is a Θ potential $\Theta(x, \mu) \not\equiv \infty$ satisfying (1.2) ([10; Théorème 8]). On account of Lemma 4.1, we may assume that μ is a finite measure on $B(\xi, (4m^2)^{-1}r_0) \cap \overline{D} \setminus \{\xi\}$. Letting $\psi(t) = 1$, we apply Theorem 2.1 to

$\Theta(x, \mu)$, and find a set E' $(\Theta, 1, 1, \phi_\xi)$-thin at ξ such that

$$\lim_{x\to\xi,\ x\notin E'} \Theta(x, \mu)/\Theta(x, \xi) = 0,$$

since $\phi_\xi(|x - \xi|) \sim \Theta(x, \xi)$ from Lemma 3.2 and $\rho(\phi_\xi) < 1$ from Lemma 3.7. Hence if $r > 0$ is small enough, then $E \cap B(\xi, r) \subset E'$, so that E is $(\Theta, 1, 1, \phi_\xi)$-thin at ξ. On the other hand, it follows from Theorem 2.2 that if E is $(\Theta, 1, 1, \phi_\xi)$-thin at ξ, then there is a finite measure μ on $\overline{D} \cap B(\xi, (4m^2)^{-1}r_0) \setminus \{\xi\}$ satisfying (1.2). Obviously $\Theta(\cdot, \mu) \not\equiv \infty$. Therefore E is minimally thin at ξ by [10; Théorème 8].

By virtue of Lemma 3.6 we have

$$C_{\Theta,1}(E(\xi, j); I_j^*(\xi)) \leqq A\tau_\xi(s^j)^2 C_{U_2,1}(E(\xi, j); I_j^*(\xi)),$$

where $U_2(x, y) = |x - y|^{2-n}$. We observe that $C_{U_2,1}$ coincides with the Newtonian capacity up to a multiplicative constant. Hence Proposition 2.1 and Remark 2.1 imply that if E is thin at ξ in the ordinary sense, then E is $(\Theta, 1, 1, \phi_\xi)$-thin at ξ. In case when E is in a nontangential cone with vertex at ξ, we have by Lemma 3.5

$$C_{\Theta,1}(E(\xi, j); I_j^*(\xi)) \sim \tau_\xi(s^j)^2 C_{U_2,1}(E(\xi, j); I_j^*(\xi)),$$

so that E is thin at ξ in the ordianry sense if and only if E is $(\Theta, 1, 1, \phi_\xi)$-thin at ξ. Thus the theorem follows.

Remark 4.1. Let $x_0 \in D \setminus (\{z_0\} \cup B(\xi, s))$. We observe from Lemmas 3.2 and 3.3 that

(4.1) $$\hat{R}^{E(\xi,\ j)}_{K(\cdot,\ \xi)}(x_0) \sim \phi_\xi(s^j)C_{\Theta,1}(E(\xi,\ j);\ I^*_j(\xi)).$$

Hence a set $E \subset D$ is minimally thin at ξ if and only if

$$\Sigma^\infty_{j=1}\ \hat{R}^{E(\xi,\ j)}_{K(\cdot,\ \xi)}(x_0) < \infty.$$

If $\lim_{j\to\infty} \hat{R}^{E(\xi,\ j)}_{K(\cdot,\ \xi)}(x_0) = 0$, then E is said to be minimally semithin at ξ (Hunt and Wheeden [4; p. 522]). It follows from (4.1) that E is minimally semithin at ξ if and only if E is $(\Theta,\ 1,\ \phi_\xi)$-semithin at ξ. The relationship between $C_{\Theta,1}$ and $C_{U_2,1}$ in the proof of Theorem 1.1 yields that if E is semithin at ξ in the ordinary sense then E is minimally semithin at ξ; in case when E is contained in a nontangential cone with vertex at ξ, E is semithin at ξ in the ordinary sense if and only if E is minimally semithin at ξ.

Remark 4.2 (cf. [11; Theorem 1]). By Theorem 2.1 we have the following: Let ψ be moderate, $\rho(\psi) < 1$, $\rho(\phi_\xi/\psi) < 1$, and μ be a measure on $\overline{D} \setminus \{z_0\}$ such that $\Theta(x,\ \mu) \not\equiv \infty$. We set $v(x) = \phi_\xi(|x - \xi|)^{-1}\psi(|x - \xi|)\Theta(x,\ \mu)$. If Γ is a nontangential cone with vertex at ξ, then the succeeding four statements are equivalent:

(i) $\lim_{r\to 0} \psi(r)\mu(B(\xi,\ r)) = 0$.

(ii) $v(x)$ has minimal semifine limit zero at ξ, i.e., there is a set E minimally semithin at ξ such that

$$\lim_{x\to\xi,\ x\in D\setminus E} v(x) = 0.$$

(iii) $v(x)$ has nontangential limit zero in L^p at ξ for

$1 \leqq p < n/(n-2)$, i.e., for each Γ

$$\lim_{r\to 0} r^{-n}\int_{\Gamma\cap B(\xi,\, r)} v(x)^p dx = 0.$$

(iv) There is a sequence $\{x_j\}_{j=1}^{\infty}$ converging to ξ such that $\{x_j\} \subset \Gamma$ for some Γ, $|x_j - \xi|/|x_{j+1} - \xi|$ is bounded, and $\lim_{j\to\infty} v(x_j) = 0$.

Now we give a local version of Theorem 1.2.

Proposition 4.1. Let $p \geqq 1$, $\rho(\hat{\phi}_{\xi p}) < 1$, ξ, ψ, μ and v be as in the above remark. Suppose that μ satisfies

(4.2) $$\lim_{r\to 0} \psi(r)\mu(B(\xi,\, r)) = 0,$$

(4.3) $$\int_\Gamma \psi(|x-\xi|)d\mu(x) < \infty, \qquad \text{if } p = 1$$
$$\int_\Gamma \hat{\psi}_p(|x-\xi|)^p f(x)^p dx < \infty,\ d\mu(x) = f(x)dx, \quad \text{if } p > 1$$

for all nontangential cones Γ with vertex at ξ. Then there exists a set $E_\xi^{\sim} \subset N(\xi)$ such that $B_{2,p}(E_\xi^{\sim}) = 0$ and if $\alpha \in N(\xi) \setminus E_\xi^{\sim}$, then

(4.4) $$\lim_{r\to 0} v(\xi + r\alpha) = 0.$$

Here, $N(\xi)$ is the totality of all nontangential unit vectors at ξ and $B_{2,p}$ is the Bessel capacity of index $(2, p)$.

If in addition, $2p > n$, then $v(x)$ has nontangential limit zero at ξ.

We postpone the proof of Proposition 4.1, and prove Theorem 1.2 first.

Proof of Theorem 1.2. It is sufficient to prove the theorem for $\xi \in S_0$ provided supp $f \subset B(\xi_0, r_0) \cap \overline{D}$. Since $G(x, f) \not\equiv \infty$, $g(y)f(y)$ is integrable. Let $\psi(t) = t^{1-n}$. We have by (1.3) $F_1 = \{\xi \in \partial D;$ either (4.2) or (4.3) does not hold for $\xi\}$ has $(n - 1)$-dimensional Hausdorff measure zero. On the other hand Dahlberg [3; Theorem 1] proved that $F_2 = \{\xi \in \partial D; \partial g/\partial n_\xi$ does not exists or vanishes$\}$ has $(n - 1)$-dimensional Hausdorff measure zero. Since $\tau_\xi(r) \sim r$ for $\xi \in \partial D \setminus F_2$, the theorem follows from Proposition 4.1 with $F = F_1 \cup F_2$.

Now the proof of Proposition 4.1 remains. For this purpose we need

Lemma 4.2. Let Γ and Γ' be nontangential cones with vertex at ξ such that $\Gamma' \supset \overline{\Gamma} \setminus \{\xi\}$. If μ satisfies (4.2) and (4.3) for Γ', then there exists a set E $(U_2, p, p, t^{2-n/p})$-thin at ξ such that

$$\lim_{x\to\xi,\ x\in\Gamma\setminus E} v(x) = 0. \tag{4.5}$$

Proof. Let $\Theta'(x, y) = \Theta(x, y)$ if $y \in \Gamma'$ and $\Theta'(x, y) = 0$ otherwise. Put $\Theta'' = \Theta - \Theta'$. On account of Theorem 2.1, Lemma 3.3 and (4.3), we can take a set E $(U_2, p, p, t^{2-n/p})$-thin at ξ such that

$$\lim_{x\to\xi,\ x\in D\setminus E} \phi_\xi(|x - \xi|)^{-1}\psi(|x - \xi|)\Theta'(x, \mu) = 0.$$

Similarly, we find a set E' $(\Theta'', 1, \phi_\xi)$-semithin at ξ such that

$$\lim_{x\to\xi,\ x\in D\setminus E'} \phi_\xi(|x - \xi|)^{-1}\psi(|x - \xi|)\Theta''(x, \mu) = 0.$$

It is sufficient to prove that $\Gamma \cap E' \cap B(\xi, r)$ is empty if $r > 0$ is small enough. In view of Lemma 3.4, we have

$$C_{\Theta'',1}(\{x\};\ I_j^*(\xi)) \geqq A\phi_\xi(s^j)^{-1} \quad \text{for } x \in \Gamma \cap I_j(\xi),$$

where A is independent of j. Since E' is $(\Theta'', 1, \phi_\xi)$-semithin at ξ, $\Gamma \cap E' \cap B(\xi, r) = \emptyset$ if r is small enough. The lemma follows.

Proof of Proposition 4.1. Without loss of generality, we may assume that $\xi = 0$. Let $L = \Gamma \cap \{x;\ |x| = 1\}$. It suffices to show that there is a set $E^{\sim} \subset L$ such that $B_{2,p}(E^{\sim}) = 0$ and (4.4) holds for all $\alpha \in L \setminus E^{\sim}$. In view of Lemma 4.2, we find a set E $(U_2, p, p, t^{2-n/p})$-thin at ξ such that (4.5) holds. Let $E(j)^{\sim} = \{x;\ |x| = 1$ and $rx \in E(j)$ for some $r > 0\}$ and $E^{\sim} = \limsup_{j\to\infty} E(j)^{\sim}$. Since $B_{2,p}(E(j)^{\sim}) \leqq As^{j(2p-n)}B_{2,p}(E(j))$ ([8; p. 116]), we have

$$\Sigma_{j=1}^{\infty} B_{2,p}(E(j)^{\sim}) < \infty,$$

so that $B_{2,p}(E^{\sim}) = 0$. Obviously (4.4) holds for all $\alpha \in L \setminus E^{\sim}$.

In case $2p > n$, letting $E(j)' = s^{-j}E(j)$, we observe that

$$\Sigma_{j=1}^{\infty} B_{2,p}(E(j)') < \infty.$$

Since $B_{2,p}(F) > 0$ if $F \neq \emptyset$, at most finite number of $E(j)'$ are not empty, which implies that

$$\lim_{x\to 0,\ x\in\Gamma} v(x) = 0.$$

Hence v(x) has nontangential limit zero at 0. The proof is complete.

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