数理解析研究所講究録
503巻 1983年 84-92

On the Asymptotic Behaviors of the Generalized Spherical Functions on Semisimple Lie Groups

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1. INTRODUCTION. This is an abstract note of [6]. Though the main result of this note is correct for more general Lie groups, called of class $\mathcal{H}$, than semisimple Lie groups, for simplicity we restrict ourselves to semisimple case. So we now assume that G is a connected semisimple Lie group of the non-compact type with finite center. Let $G=KAN$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ the corresponding Iwasawa decompositions of G and its Lie algebra $\mathfrak{g}$.

The Eisenstein integrals, that is the matrix elements of representations of principal series for G, play an essential role in harmonic analysis on G. Therefore it is very important to know their asymptotic behaviors. In fact, the leading terms of the asymptotic expansions of them are the Harish-Chandra C-functions and, as is well known, they relate closely with the Plancherel measure on G[8,9,10,11]. Moreover, we need to know their behaviors of higher order to carry out further analysis on G. In this note we focus our attention on the Harish-Chandra expansions of Eisenstein integrals and their coefficients. For the zonal spherical function $\phi_\nu(x) = \int_K e^{(\nu-\rho)(H(xk)}dk$ $(x \in G)$, when $x = h$ varying in the positive Weyl chamber A^+ of A, $\phi_\nu(h)$ is expanded into an infinite series by Harish-Chandra[7] as follows:

$$\phi_\nu(h) = e^{-\rho(\log h)} \sum_{s\in W} c(s\nu)\Phi(s\nu:h),$$

$$\Phi(\nu:h) = \sum_{\lambda\in L} \Gamma_\lambda(\nu-\rho)e^{(\nu-\rho)(\log h)} \qquad (h \in A^+).$$

Here $c(\ )$ is the Harish-Chandra c-function and Γ_λ $(\lambda \in L)$ are the coefficients. In his paper [14] Gangolli gave a remarkable estimate for these coefficients. The purpose of this note is to give the Gangolli estimate for **the coefficients of the Harish-Chandra expansions of the Eisenstein integrals.**

2. PRELIMINARIES. Let M be the centralizer of A in K. Denote by F_R and F_C the real dual space of $\mathfrak{a}$ and its complexification, respectively. Write $F = (-1)^{1/2}F_R$. Let $\tau = (\tau_1,\tau_2)$ be a double unitary representation of K on a finite dimensional Hilbert space V. Put

$$V_M = \{v \in V;\ \tau_1(m)v = v\tau_2(m) \quad \text{for any} \quad m \in M\}.$$

Then the following integral is called the Eisenstein integral or the generalized spherical function:

$$E(v:\nu:x) = \int_K \tau_1(\kappa(xk))v\tau_2(k^{-1})e^{(\nu-\rho)(H(xk))}dk. \tag{1}$$

Let ω be the Casimir operator. Then E satisfies the following differential equation:

$$E(v:\nu:x;\omega) = \{\langle\nu,\nu\rangle - \langle\rho,\rho\rangle + \tau_2(\omega_{\mathfrak{m}})\}E(v:\nu:x). \tag{2}$$

Here $\omega_{\mathfrak{m}}$ denotes the Casimir operator on M.

Let θ be the Cartan involution of $\mathfrak{g}$ with respect to $\mathfrak{k}$. Let $\mathfrak{s}$ be the subspace of all $X \in \mathfrak{g}$ such that $\theta(X)=-X$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{a}\subset\mathfrak{h}$ and put $\mathfrak{h}_{\mathfrak{k}}=\mathfrak{h}\cap\mathfrak{k}$. Let Σ denote the set of all roots of $(\mathfrak{g}, \mathfrak{h})$ and $\Sigma_0=\{\alpha_1,\dots,\alpha_\ell\}$ the set of all simple roots in Σ. We consider the lexicographic order in F_R defined by $\alpha_1,\dots,\alpha_\ell$ and fix a compatible order in the dual space of $\mathfrak{h}^*=\mathfrak{a}+(-1)^{1/2}\mathfrak{h}_{\mathfrak{k}}$. Let Δ_+ denote the set of positive roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$ such that $\tilde{\alpha}=\alpha|_{\mathfrak{a}}\neq 0$. For each $\alpha\in\Delta_+$ define the element $Q_{\tilde{\alpha}}\in\mathfrak{a}$ so that $\tilde{\alpha}(H)=\langle Q_{\tilde{\alpha}},H\rangle$ for all $H\in\mathfrak{a}$. For each $\alpha\in\Delta_+$, choose the root vectors $X_{\pm\alpha}\in\mathfrak{g}_C^{\pm\alpha}$ so that $B(X_\alpha, X_{-\alpha})=1$, B denoting the Killing form, and write them as $X_{\pm\alpha}=Y_{\pm\alpha}+Z_{\pm\alpha}$ $(Y_{\pm\alpha}\in\mathfrak{k}_C$, $Z_{\pm\alpha}\in\mathfrak{p}_C)$.

The following lemma gives the radial part of the Casimir operator.

<u>Lemma 1. Denote the radial part of</u> ω <u>(resp.</u> $\omega_{\mathfrak{m}}$<u>) by</u> $\mathcal{R}(\omega)$ <u>(resp.</u> $\mathcal{R}(\omega_{\mathfrak{m}})$<u>). Then we have</u>

$$(3)\qquad \mathcal{R}(\omega) = \mathcal{R}(\omega_{\mathfrak{m}}) + \sum_{i=1} H_i^2 + \sum_{\alpha\in\Delta_+} \coth(\alpha)Q_{\tilde{\alpha}}$$
$$- 2\sum_{\alpha\in\Delta_+} (\mathrm{sh}(\alpha))^{-2}\{1\otimes 1\otimes Y_\alpha Y_{-\alpha} + Y_\alpha Y_{-\alpha}\otimes 1\otimes 1\}$$
$$+ 4\sum_{\alpha\in\Delta_+} (\mathrm{sh}(\alpha))^{-1}\coth(\alpha)(Y_\alpha\otimes 1\otimes Y_{-\alpha}).$$

3. THE HARISH-CHANDRA EXPANSION AND THE MAIN THEOREM. Put

$$L = \{\lambda=n_1\alpha_1+\cdots+n_\ell\alpha_\ell,\quad n_i\in Z_+,\ i=1,\ldots,\ell\}.$$

If $\lambda = n_1\alpha_1+\cdots+n_\ell\alpha_\ell\in L$, $m(\lambda) = n_1+\cdots+n_\ell$ is called its level. Let γ be the endomorphism of $\mathrm{Hom}_C(V_M, V_M)$ defined by

$$\gamma(T) = [\tau_2(\omega_{\mathfrak{m}}), T],\qquad T\in\mathrm{Hom}_C(V_M, V_M).$$

Let $\gamma_1,\ldots,\gamma_t$ be the set of all distinct eigenvalues of γ with multiplicities $m_1,\ldots,m_t$, respectively. It is known that they are all real. We assume that $\gamma_1<\cdots<\gamma_t$. We review the definition of Γ_λ $(\lambda\in L)$. Let $\Gamma_0\equiv 1$. For $\lambda\neq 0$, let Γ_λ be the function on F_C with values in $\mathrm{Hom}_C(V_M, V_M)$ given by the following recursion formula:

$$\{2\lambda - \langle\lambda, \lambda - 2\rho\rangle\}\Gamma_\lambda - \gamma(\Gamma_\lambda) = 2\sum_{\alpha\in\Delta_+}\sum_{n\geq 1}\{\tilde{\alpha} - \langle\tilde{\alpha}, \lambda - 2n\tilde{\alpha}\rangle\}\Gamma_{\lambda-2n\tilde{\alpha}} \tag{4}$$

$$+ 8\sum_{\alpha\in\Delta_+}\sum_{n\geq 1}(2n-1)\{\tau_1(Y_\alpha)\tau_2(Y_{-\alpha})\}\Gamma_{\lambda-(2n-1)\tilde{\alpha}}$$

$$- 8\sum_{\alpha\in\Delta_+}\sum_{n\geq 1} n\{\tau_1(Y_\alpha Y_{-\alpha}) + \tau_2(Y_\alpha Y_{-\alpha})\}\Gamma_{\lambda-2n\tilde{\alpha}},$$

where $\Gamma_\lambda \equiv 0$ if $\lambda \notin L$.

Denote by L' the set of $\lambda\neq 0$ in L. For each i $(1 \leqslant i \leqslant t)$ and $\lambda \in L'$ put

$$\sigma_{\lambda,i} = \{\nu \in F_C;\ 2\langle\lambda,\nu\rangle = \langle\lambda,\lambda\rangle + \gamma_i\}.$$

Let T denote the compliment of the set $\bigcup_{\lambda\varepsilon L'}\bigcup_i \sigma_{\lambda,i}$ in F_C. Let T' be the set of all $\nu \in F'_C$ such that $w\nu \in T$ for all $w \in W$, W denoting the Weyl group of (G, A).

Theorem 2. (Harish-Chandra).

(i) For a fixed $\nu \in T$,

$$h \to \Phi(\nu:h) = \sum_{\lambda\varepsilon L}\Gamma_\lambda(\nu-\rho)h^{\nu-\lambda} \quad \text{is analytic on } A^+.$$

(ii) $$\Phi(\nu:h;e^\rho\circ\mathcal{R}(\omega)\circ e^{-\rho}) = (\langle\nu,\nu\rangle - \langle\rho,\rho\rangle + \tau_2(\omega_{\mathfrak{m}}))\Phi(\nu:h).$$

(iii) $$h^\rho E(v:\nu:h) = \sum_{w\in W}\Phi(w\nu:h)C(w:\nu)v, \quad \nu \in T',$$

where $C(w:\nu)$ $(w \in W)$ are certain meromorphic functions on F_C with values in $\mathrm{Hom}_C(V_M, V_M)$.

Fix $a > 0$ and put

$$R(a) = \{\xi+\eta\in L';\ \xi \in F,\ \eta \in F_R,\ -\eta+a\rho \in \mathrm{Cl}(F_R^+)\}$$

We want to know the behavior of Γ_λ in the cone $R(a)$ and consider the following finite set (may be empty):

$$L_1'(a) = \{\lambda \in L' : -\langle\lambda,\lambda\rangle + 2a\langle\lambda,\rho\rangle - \gamma_1 \geqslant 0\}.$$

which is the set of $\lambda \in L'$ such that the determinant of the coefficient of Γ_λ in (4) takes value 0 in $R(a)$. Put

$$p_\lambda(\nu) \equiv 1 \quad \text{if} \quad \lambda \notin L_1'(a);$$

$$p_\lambda(\nu) = \prod_{\substack{1\leqslant i\leqslant t \\ d(a:\lambda)\leqslant -\gamma_i}} (2\langle\lambda,\nu\rangle - \langle\lambda,\lambda\rangle - \gamma_i)^{m_i},$$

$$d'(\lambda) = \sum_{1\leqslant i\leqslant t,\ d(a:\lambda)\leqslant-\gamma_i} m_i$$

for $\lambda \in L_1'(a)$, where $d(a:\lambda)=\langle\lambda,\lambda\rangle-2a\langle\lambda,\rho\rangle$. If $\lambda, \lambda' \in L$ and $\lambda-\lambda' \in L$ then we denote it by $\lambda \gg \lambda'$. We also put

$$P(\nu) = \prod_{\lambda\in L_1'(a)} p_\lambda(\nu), \qquad d = \sum_{\lambda\in L_1'(a)} d'(\lambda) < +\infty \ ;$$

$$P_\lambda(\nu) = \prod_{\substack{\lambda'\in L' \\ \lambda'\ll\lambda}} p_{\lambda'}(\nu), \qquad d(\lambda) = \sum_{\substack{\lambda'\in L' \\ \lambda'\ll\lambda}} d'(\lambda)$$

for $\lambda \in L_1'(a)$. Then as is easily seen, all singularities of Γ_λ in the domain $R(a)$ concentrate on the polynomial $P(\lambda)$. The following result is the main theorem.

<u>Theorem 3.</u> <u>There exist constants</u> $D, d_1>0$, <u>depending only on</u> τ, <u>which satisfy</u>

$$\|P_\lambda(\nu)\Gamma_\lambda(\nu-\rho)\| \leqslant D(1 + |\nu| + m(\lambda))^{2d} m(\lambda)^{d_1}$$

<u>uniformly in</u> $\lambda\in L'$, $\nu\in R(a)$.

4. A SKETCH OF THE PROOF OF THE THEOREM. We give in this section a sketch of the proof. We need the following lemma.

Lemma 4. Put $H=\log h$ $(h \in A^+)$ and $\Delta(h) = h^{2\rho}\prod_{\alpha\in\Delta_+}(1-h^{-2\alpha})$ $(h\in A)$. Then we have

$$(5)\qquad \Delta(h)^{1/2}\circ\mathcal{R}(\omega)\circ\Delta(h)^{-1/2} = \mathcal{R}(\omega_{\mathfrak{m}}) + \sum_{i=1}^{\ell} H_i^2 - \langle\rho,\rho\rangle$$

$$+ \sum_{\alpha\in\Delta_+}\langle\tilde{\alpha},\tilde{\alpha}\rangle\sum_{j\geq 1} j e^{-2j\alpha(H)} - \sum_{\substack{\alpha,\beta\in\Delta_+\\ \alpha\neq\beta}}\langle\tilde{\alpha},\tilde{\alpha}\rangle\sum_{\substack{j\geq 1\\ k\geq 0}} e^{-2j\alpha(H)-2k\beta(H)}$$

$$- 8\sum_{\alpha\in\Delta_+}\sum_{j\geq 1} j e^{-2j\alpha(H)}(1\otimes 1\otimes Y_\alpha Y_{-\alpha} + Y_\alpha Y_{-\alpha}\otimes 1\otimes 1)$$

$$+ 8\sum_{\alpha\in\Delta_+}\sum_{j\geq 0}(2j+1)e^{-(2j+1)\alpha(H)}(Y_\alpha\otimes 1\otimes Y_{-\alpha}).$$

The most important thing is that $\Delta(h)^{1/2}\circ\mathcal{R}(\omega)\circ\Delta(h)^{-1/2}$ in the lemma is an operator of the Sturm-Liouville type. If we consider the function Ψ given by the following in stead of Φ itself:

$$\Psi(\nu:h) = \Delta(h)^{1/2}h^{-\rho}\Phi(\nu:h) \qquad h\in A^+,$$

then Ψ satisfies the following differential equation:

$$(6)\qquad (\Delta(h)^{1/2}\circ\mathcal{R}(\omega)\circ\Delta(h)^{-1/2})T = (\langle\nu,\nu\rangle - \langle\rho,\rho\rangle + \tau_2(\omega_{\mathfrak{m}}))\Psi.$$

Expand Ψ into the series

$$\Psi(\nu:h) = h^{\nu}\sum_{\lambda\in L} a_\lambda(\nu)h^{-\lambda} \qquad h\in A^+.$$

Then, using Lemma 4, we obtain the recursion formula for $a_\lambda(\nu)$:

$$(7)\qquad [2\langle\lambda,\nu\rangle - \langle\lambda,\lambda\rangle]a_\lambda(\nu) - \gamma(a_\lambda(\nu))$$

$$= \sum_{\alpha\in\Delta_+} [\langle\tilde{\alpha},\tilde{\alpha}\rangle - 8F_\alpha] \sum_{j\geqslant 1} j a_{\lambda-2j\tilde{\alpha}}(\nu) - \sum_{\substack{\alpha,\beta\in\Delta_+\\ \alpha\neq\beta}} \langle\tilde{\alpha},\tilde{\beta}\rangle \sum_{\substack{j\geqslant 1\\ k\geqslant 0}} a_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(\nu)$$

$$+ 8\sum_{\alpha\in\Delta_+} G_\alpha \sum_{j\geqslant 1} (2j-1) a_{\lambda-(2j-1)\tilde{\alpha}}(\nu),$$

where F_α and G_α are defined by

$$F_\alpha = \tau_1(Y_\alpha Y_{-\alpha}) + \tau_2(Y_\alpha Y_{-\alpha}); \qquad G_\alpha = \tau_1(Y\ )\circ\tau_2(Y_{-\alpha}).$$

We pay attention to the fact that all singularities of a_λ in the domain $R(a)$ are concentrated upon P_λ and put

$$Q_\lambda(\nu) = P_\lambda(\nu)(1 + |\nu| + |\lambda|)^{-2d(\lambda)} \quad\text{and}\quad q_\lambda(\nu) = p_\lambda(\nu)(1 + |\nu| + |\lambda|)^{-2d'(\lambda)}.$$

Moreover, we define $b_\lambda(\nu)$ for all $\lambda \in L$ by

$$b_0(\nu) \equiv 1 \qquad \text{if } \lambda = 0;$$

$$b_\lambda(\nu) = Q_\lambda(\nu)a_\lambda(\nu) \qquad \text{if } \lambda \in L'.$$

Then we obtain the following recursion formula for $b_\lambda(\nu)$. We put $\gamma(\lambda:\nu) = (2\langle\lambda,\nu\rangle - \langle\lambda,\lambda\rangle)I - \gamma$.

(8) $$\gamma(\lambda:\nu)b_\lambda(\nu) = \sum_{\alpha\in\Delta_+} [\langle\tilde{\alpha},\tilde{\alpha}\rangle - 8F_\alpha] q_\lambda(\nu) \sum_{j\geqslant 1} Q^1_{\lambda,j}(\nu) b_{\lambda-2j\tilde{\alpha}}(\nu)$$

$$- \sum_{\substack{\alpha,\beta\in\Delta_+\\ \alpha\neq\beta}} \langle\tilde{\alpha},\tilde{\beta}\rangle q_\lambda(\nu) \sum_{\substack{j\geqslant 1\\ k\geqslant 0}} Q_{\lambda,j,k}(\nu) b_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(\nu)$$

$$+ 8\sum_{\alpha\in\Delta_+} G_\alpha q_\lambda(\nu) \sum_{j\geqslant 1} Q^2_{\lambda,j}(\nu) b_{\lambda-(2j-1)\tilde{\alpha}}(\nu).$$

Where the polynomials $Q^1_{\lambda,j}$, $Q_{\lambda,j,k}$ and $Q^2_{\lambda,j}$ are given by the relation

$$Q^1_{\lambda,j}(\nu)Q_{\lambda-2j\tilde{\alpha}}(\nu) = Q_{\lambda,j,k}(\nu)Q_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(\nu) = Q^2_{\lambda,j}(\nu)Q_{\lambda-(2j-1)\tilde{\alpha}}(\nu) = Q_\lambda(\nu)q_\lambda(\nu)^{-1}$$

An argument parallel to [14] leads to our assertion. For more detail, see [6].

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