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<td>著者</td>
<td>Mukai, Juno</td>
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<tr>
<td>引用</td>
<td>数理解析研究所講究録 (1983), 505: 32-45</td>
</tr>
<tr>
<td>発行日</td>
<td>1983-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/103733">http://hdl.handle.net/2433/103733</a></td>
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<tr>
<td>タイプ</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>テキストバージョン</td>
<td>publisher</td>
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The Order of the Attaching Class of the Suspended Quaternionic Quasi-Projective Space

By

Juno Mukai

§ 0. Introduction

In this note, \( F \) denotes the field of the complex numbers \( \mathbb{C} \) or the field of the quaternions \( \mathbb{H} \). We denote by \( \mathbb{P}^n \) the \( F \)-projective space of \( n \) \( F \)-dimensions and by \( Q_n(F) \) the quasi-\( F \)-projective space. \( G_n(F) \) denotes the unitary group \( U(n) \) or the symplectic group \( Sp(n) \) according as \( F \) is \( \mathbb{C} \) or \( \mathbb{H} \). Let \( d \) be the dimension of \( F \) over the field of the real numbers \( \mathbb{R} \) and \( S^{dn-1} \) the unit sphere in \( F^n \). Let \( T'_n : S^{dn-2} \rightarrow G_{n-1}(F) \) be the characteristic map for the normal form of the principal \( G_{n-1}(F) \)-bundle over \( S^{dn-1} \). Then, as is well known ( [2], [3] and [9]), \( \operatorname{Im} T'_n = Q_{n-1}(F) \), precisely, the following diagram commutes:

\[
\begin{array}{ccc}
S^{dn-2} & \xrightarrow{T'_n} & G_{n-1}(F) \\
& S^{dn-2} \xrightarrow{T_n} & Q_{n-1}(F) \\
& T'_n \downarrow & \downarrow j_{n-1} \\
& G_{n-1}(F) & \\
\end{array}
\]

where \( j_{n-1} \) is the canonical reflection map. \( Q_n(F) = Q_{n-1}(F) \cup T'_n e^{dn-1} \) and \( Q_n(C) = E(CP^{n-1}) \), where \( E(\cdot) \) denotes the reduced suspension and \( CP^{n-1} \) a disjoint union of \( CP^{n-1} \) and \{ one point \}.

Let \( \omega_n = \omega_n(F) \) be the homotopy class of \( T_n \) and \( p : Q_n(C) \rightarrow Q_n(C)/Q_1(C) = ECP^{n-1} \) the collapsing map. In the previous paper [6], we proved that the \( k \)-th suspension \( E^k(p_* \omega_n(C)) \) is of order \( n! \) for \( k \geq 0 \).

The purpose of this note is to examine the order of \( E^k \omega_{n-1}(H) \).
Let $\alpha$ be an element of a homotopy group $\pi_n(\ )$ and $E^\infty \alpha \in \pi_n(\ )$ the stable element of $\alpha$. $o(\beta)$ denotes the order of $\beta$. Then, our result is the following

Theorem. i). $o(E^k \omega_{n-1}(H)) = 2 \cdot (2n - 1)!$ for $k \geq 0$ if $n$ is even.

ii). $o(E^\infty \omega_{n-1}(H)) = (2n - 1)!$ if $n$ is odd.

Our method is essentially to use the K-theory. To examine $o(\omega_{n-1}(H))$, we use the Toda's theorem about the generator of $\pi_{2n-1}(U(n))$ [6] and the group structure of $\pi_{4n+2}(Sp(n))$ [4]. To determine the lower bound of $o(E^k \omega_{n-1}(H))$, we use the standard method of D. M. Segal [7] from the unstable viewpoint, exactly, we use the Hurwicz homomorphism $h: \pi_{k+4n-1}(E^k \Omega_n(H)) \rightarrow H_{k+4n-1}(E^k \Omega_n(H); \mathbb{Z})$. A powerful tool is the Toda-Kozima's map $\xi_n: \Omega_n(H) \rightarrow Q_{2n}(C)$ [8].

Our result overlaps partially with the works of K. Morisugi [5] and G. Walker [9].

The author wishes to express his sincere gratitude to Professor S. Sasao for many advices given during the preparation of this paper.

§1. Determination of $o(\omega_{n-1}(H))$ for even $n$

First we recall the definition of the quasi-projective space and the reflection map. $S(F^n)$ denotes the unit sphere in $F^n$. $Q_n(F)$ is the space obtained from $S(F^n) \times S(F)$ by imposing the equivalence relation: $(u, q) \sim (ug, g^{-1}qg)$ for $g \in S(F)$ and collapsing $S(F^n) \times \{1\}$ to a point. The reflection map $j_n = j_n(F): Q_n(F) \rightarrow G_n(F)$ is defined as follows:

$$j_n([u, q])(v) = v + u(q - 1)u, v$$

for $u \in S(F^n)$, $q \in S(F)$ and $v \in F^n$, where $<u, v> = \sum_{k=1}^{n} u_k v_k$ for $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. 

Let $z = x + jy \in H$, where $x, y \in C$. By regarding $x \in C$ as $x + j0 \in H$, we have the injection $C \to H$. Obviously, this induces the canonical maps $i_n: Q_n(C) \to Q_n(H)$ and $i'_n: U(n) \to \text{Sp}(n)$. From the definition, the following diagram commutes:

$\begin{array}{ccc}
Q_n(C) & \xrightarrow{i_n} & Q_n(H) \\
\downarrow j_n & & \downarrow j_n \\
U(n) & \xrightarrow{i'_n} & \text{Sp}(n).
\end{array}$

(1.1)

In the complex case, we can define the reduced reflection map [6]:

$j_n = j_n(C): \mathbb{CP}^{n-1} \cong Q_n(C)/Q_1(C) \to U(n)/U(1) \cong \text{SU}(n).

By abuse of notation, we often use the same letter $j_n$ for the reduced case.

Lemma 1.2. i). If $n$ is even, $j_n^*: \pi_{4n-1}(Q_n(H)) \to \pi_{4n-1}(\text{Sp}(n))$ is an epimorphism.

ii). If $n$ is odd, $\text{Im} j_n^* = a\pi_{4n-1}(\text{Sp}(n))$, where $a = 1$ or 2.

Proof. Let $p: Q_{2n}(C) \to Q_{2n}(C)/Q_1(C) \cong \mathbb{CP}^{2n-1}$ be the collapsing map, $k: Q_n(H) \to Q_{2n}(H)$ and $k': \text{Sp}(n) \to \text{Sp}(2n)$ the inclusion maps, respectively. Then, by (1.1), the following diagram commutes for $r = 4n - 1$:

$\begin{array}{cccc}
\pi_r(\mathbb{CP}^{2n-1}) & \xleftarrow{p^*} & \pi_r(Q_{2n}(C)) & \xrightarrow{i_{2n}} & \pi_r(Q_{2n}(H)) & \xleftarrow{k^*} & \pi_r(Q_n(H)) \\
| & \downarrow j_{2n}(C)^* & & \downarrow j_{2n}(C)^* & & \downarrow j_{2n}^* & \downarrow j_n^* \\
\pi_r(\text{SU}(2n)) & = & \pi_r(U(2n)) & \xrightarrow{i_{2n}^*} & \pi_r(\text{Sp}(2n)) & \xleftarrow{k'^*} & \pi_r(\text{Sp}(n)).
\end{array}$

$p^*$ is an epimorphism since $Q_{2n}(C) \cong \mathbb{CP}^{2n-1} \vee S^1$. By Theorem 4.1 of [6], $j_{2n}(C)^*$ is an epimorphism. So, $j_{2n}(C)^*$ is an epimorphism. $k^*$ and $k'^*$ are isomorphisms respectively. As is well known, $i_{2n}^*$ is an isomorphism if $n$ is even and $\text{Im} i_{2n}^* = 2\pi_{4n-1}(\text{Sp}(2n))$ if $n$ is odd. Therefore, the above commutative diagram leads us to the assertion. This completes the proof.
Proposition 1.3. i). \( o(\omega_{n-1}) = 2 \cdot (2n - 1)! \) for even \( n \).

ii). \( o(\omega_{n-1}) = \frac{(2n - 1)!}{a} \) for odd \( n \), where \( a \) is the same number as in Lemma 1.2.

Proof. Let \( p: (Q_n(H), Q_{n-1}(H)) \to (S^{4n-1}, \star) \) be the collapsing map. We consider the natural homomorphism between the exact sequences for \( r = 4n - 1 \):

\[
\begin{array}{cccccc}
\pi_r(Q_n(H)) & \xrightarrow{j_n^*} & \pi_r(Q_n(H), Q_{n-1}(H)) & \xrightarrow{\partial} & \pi_{r-1}(Q_{n-1}(H)) & \longrightarrow \\
\downarrow{j_{n-1}^*} & & \downarrow{p^*} & & \downarrow{j_{n-1}^*} & \\
\pi_r(Sp(n)) & \xrightarrow{p^*} & \pi_r(S^{4n-1}) & \xrightarrow{\Delta'} & \pi_{r-1}(Sp(n-1)) & \longrightarrow \\
\end{array}
\]

where the mappings are canonical and \( \partial \) and \( \Delta' \) are the connecting homomorphisms.

As is well known, \( \pi_{4n-1}(Sp(n)) \cong \mathbb{Z} \), \( \pi_{4n-2}(Sp(n)) \cong 0 \) and \( \pi_m(S^n) = \{1_m\} \cong \mathbb{Z} \). By the Blakers-Massey theorem [1], \( p_* \) is an isomorphism. By the definition, \( \omega_{n-1} = \Delta'(1_{4n-1}) \), where \( \Delta = \partial \circ \partial^{-1} \). So, by Theorem 2.2 of [4], \( j_{n-1}^* \) is an epimorphism and the following holds:

(1.4) \( \pi_{4n-2}(Sp(n-1)) = \{j_{n-1}^*\omega_{n-1}\} \cong \mathbb{Z} \cdot (2n-1)! \), where \( b = 1 \) for odd \( n \) and \( b = 2 \) for even \( n \).

By the exactness of the upper sequence, \( o(\omega_{n-1}) \) is equal to the order of the cokernel of \( j_n^* \). Hence, by (1.4), Lemma 1.2 and by the above commutative diagram, we have the assertion. This completes the proof.

By inspecting the above proof, we have the following

Proposition 1.5. \( j_n^*: \pi_{4n-1}(Q_n(H)) \to \pi_{4n-1}(Sp(n)) \) is an epimorphism if and only if \( o(\omega_{n-1}) = b \cdot (2n - 1)! \), where \( b \) is the same number as in (1.4).
2. Some fundamental facts

For $n \geq 0$, $X_n$ denotes a connected finite CW complex such that $X_0 = \{+\}$ and $X_n = e^0 \cup e^1 \cup \ldots \cup e^n$ for $n \geq 1$. Here $r = r_n = d_n - \epsilon$ with $\epsilon = 0$ or $1$ and $d - \epsilon \geq 2$. $\theta_{n-1}: S^{r-1} \rightarrow X_{n-1}$ denotes the attaching map, and so $X_n = X_{n-1} \bigcup_{\theta_{n-1}} e^r$. For example, $X_n = F^{\mathbb{P}^n} (\text{ d = 2 or 4 and } \epsilon = 0)$ and $X_n = Q_n(H)$ (d = 4 and $\epsilon = 1$).

Let $p: X_n \rightarrow X_n / X_{n-1} = S^r$ and $p': (X_n, X_{n-1}) \rightarrow (S^r, +)$ be the collapsing maps. Let $\beta: \pi_{r+m}(E^m_{X_n}, E^m_{X_{n-1}}) \rightarrow \pi_{r+m-1}(E^m_{X_{n-1}})$ be the connecting homomorphism. Then, $(E^m_{p'})_*: \pi_{r+m}(E^m_{X_n}, E^m_{X_{n-1}}) \rightarrow \pi_{r+m}(S^{r+m})$ is an isomorphism for $m \geq 0$ [1], and we define a homomorphism $\Delta: \pi_{r+m}(S^{r+m}) \rightarrow \pi_{r+m-1}(E^m_{X_{n-1}})$ by the composition $\beta \circ (E^m_{p'})_*^{-1}$. By the definition, $\Delta(t_{r+m}) = E^m_{b - 1}$, where the same letter is used for a mapping and its homotopy class.

Let $h = h_m: \pi_{r+m}(E^m_{X_n}) \rightarrow H_{r+m}(E^m_{X_n}; Z) \cong Z$ for $m \geq 0$ be the Hurewicz homomorphism and $h(n, m)$ the non-negative integer such that $\text{Im } h = h(n, m) H_{r+m}(E^m_{X_n}; Z)$. Then we have the following

Lemma 2.1. $o(E^m_{b - 1}) = h(n, m)$.

Proof. $j: (E^m_{X_n}, +) \rightarrow (E^m_{X_n}, E^m_{X_{n-1}})$ denotes the inclusion. Then, we consider the commutative diagram:

$$
\begin{array}{ccc}
\pi_{r+m}(E^m_{X_n}) & \xrightarrow{j_*} & \pi_{r+m}(E^m_{X_n}, E^m_{X_{n-1}}) \\
\downarrow h & & \downarrow h' \\
H_{r+m}(E^m_{X_n}; Z) & \xrightarrow{j_*} & H_{r+m}(E^m_{X_n}, E^m_{X_{n-1}}; Z),
\end{array}
$$

where $h'$ denotes the relative Hurewicz homomorphism and the upper sequence is exact. From the cell structure of $X_n$, the lower $j_*$ is an isomorphism. By the relative Hurewicz theorem, $h'$ is an isomorphism. This completes the proof.
According to [8], a representative element of $Q_n(H)$ can be taken as 
$(x + jy, e^{it})$, where $x, y \in \mathbb{C}^n$ satisfying $x + jy \in S(H^n)$ and $0 \leq t \leq 1$. Toda and Kozima defined $\xi_n : Q_n(H) \longrightarrow Q_{2n}(C)$ by the equation

$$\xi_n[(x + jy, e^{it})] = [(x \oplus y, e^{2it})].$$

We define $t_n : Q_n(H) \longrightarrow \text{ECF}^{2n-1}$ by the composition $p \cdot \xi_n$, where $p : Q_{2n}(C) \longrightarrow \text{ECF}^{2n-1}$ is the collapsing map. From the definition, the following diagram commutes for $k < n$:

\[
\begin{array}{ccc}
Q_k(H) & \xrightarrow{t_k} & \text{ECF}^{2k-1} \\
\downarrow i & & \downarrow i' \\
Q_n(H) & \xrightarrow{t_n} & \text{ECF}^{2n-1},
\end{array}
\]

where $i$ and $i'$ are the canonical inclusions.

The following lemma is a reduced version of Proposition 2.5 of [8].

**Lemma 2.3 (Toda-Kozima).** The following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
Q_n(H) & \xrightarrow{t_n} & \text{ECF}^{2n-1} \\
\downarrow j_n & & \downarrow j_{2n} \\
\text{Sp}(n) & \xrightarrow{c} & \text{SU}(2n),
\end{array}
\]

where $c$ is the complexification map.

Let $p : Q_n(H) \longrightarrow Q_n(H)/Q_{n-1}(H) = S^{4n-1}$ for $n \geq 1$ and $p' : \text{ECF}^{2n-1} \longrightarrow \text{ECF}^{2n-1}/\text{ECF}^{2n-3} = S^{4n-3} \setminus S^{4n-1}$ for $n \geq 2$ be the collapsing maps. Then,
by (2.2), there exists a mapping \( t'_n: S^{4n-3} \to S^{4n-3} \) for \( n \geq 2 \) such that the following diagram commutes:

\[
\begin{array}{ccc}
Q_n(H) & \xrightarrow{t_n} & \text{BCF}^{2n-1} \\
\downarrow p & & \downarrow p' \\
S^{4n-1} & \xrightarrow{t'_n} & S^{4n-3} \vee S^{4n-1}.
\end{array}
\]

Let \( p_2: S^{4n-3} \vee S^{4n-1} \to S^{4n-1} \) for \( n \geq 2 \) be the projection map. Then, we have the following

Lemma 2.5. deg \( t_1 = -1 \) and deg \( (p_2 t'_n) = (-1)^n \) for \( n \geq 2 \).

**Proof.** We define \( q_n: S(H^n) \to S(C^{2n}) \) by the equation

\[
q_n(x + jy) = x \oplus y
\]

for \( x, y \in C^n \). It is clear that \( q_n \) is a homeomorphism and deg \( q_n = (-1)^n \).

By Lemma 2.3, \( t_1 = q_1 \) and \( p_2 t'_n = q_n \) for \( n \geq 2 \). This completes the proof.

Hereafter the same letter is often used for a mapping and its homotopy class. Let \( \gamma_n = \gamma_n(F): S(F^{n+1}) \to F^n \) be the projection map. Let \( i: \text{BCF}^{2n-1} \to \text{BCF}^{2n} \) be the inclusion map. Then, we have the following

Proposition 2.6. \( (-1)^{n+1} E_2 \gamma_n(C) = it \omega \) \( n(H) \).

**Proof.** By (2.2) and (2.4), the following diagram commutes for \( r = 4n + 3 \):
\[ \pi_r(S^{4n+3}) \xrightarrow{P^*} \pi_r(Q_{n+1}(H), Q_n(H)) \xrightarrow{\partial} \pi_{r-1}(Q_n(H)) \]

\[ \downarrow t_{n+1}^* \quad \downarrow t_{n+1}^* \quad \downarrow t_n^* \]

\[ \pi_r(S^{4n+1} \setminus S^{4n+3}) \xrightarrow{P'_n} \pi_r(\text{ECP}^{2n+1}, \text{ECP}^{2n-1}) \xrightarrow{\partial'} \pi_{r-1}(\text{ECP}^{2n-1}) \]

\[ \downarrow P_{2*} \quad \downarrow i_*^1 \quad \downarrow i_* \]

\[ \pi_r(S^{4n+3}) \xrightarrow{P''_n} \pi_r(\text{ECP}^{2n+1}, \text{ECP}^{2n}) \xrightarrow{\partial''} \pi_{r-1}(\text{ECP}^{2n}), \]

where the mappings are canonical.

\( p_* \) and \( p''_* \) are isomorphisms. We note that \( \omega_n(H) = \partial p_*^{-1}(14n+3) \) and \( EY_{2n}(C) = \partial''(14n+3) \). So, by Lemma 2.5 and the above commutative diagram, we have the assertion. This completes the proof.

Remark 1. Owing to Proposition 2.6, it suffices to take \((-1)^{n+1}t_\lambda \omega_n \) as in Proposition 6.5.ii) of [6]. By Theorem 1.2 of [6] and Proposition 1.3, \( o(\lambda_{2n}) = (2n + 1)! \) or \( 2^2(2n + 1)! \). In the last section, we shall show that \( o(\lambda_4) = 5! \) (cf. Lemma 11.1 of [6]).

§ 3. Determination of the lower bound of \( o(E_{2n-1}^n(H)) \).

Let \( v \in R(\text{CP}^{2n-1}) \) be the stable isomorphism class of the canonical line bundle over \( \text{CP}^{2n-1} \). We denote by \( I_C: R(W) \rightarrow R(E^2) \) the Bott periodicity isomorphism. The following Lemma is well known (cf. Lemma 2.2 of [8]).

Lemma 3.1. \( I_C(v) \in R(E^2 \text{CP}^{2n-1}) \) is represented by the adjoint of the composite of the canonical maps:

\[
\text{ECP}^{2n-1} \xrightarrow{j_{2n}} \text{SU}(2n) \xrightarrow{i} \text{U}(2n) \xrightarrow{k} \Omega \text{BU}(2n),
\]

where \( k \) is the homotopy equivalence.
Hereafter, \( \mathbb{Z} \) or the rational number field \( \mathbb{Q} \) is taken as the coefficients of the homology or cohomology groups, unless otherwise stated.

Let \( \text{ch}^k: K(\cdot) \to H^{2k}(\cdot; \mathbb{Q}) \) be the \( k \)-th Chern character and \( \text{ch} = \sum_k \text{ch}^k \) the total Chern character. Let \( \sigma: \mathbb{H}^i(\mathbb{E}) \to \mathbb{H}^{i-1}(\mathbb{E}) \) be the suspension isomorphism. Then, as is well known, the following diagram commutes:

\[
\begin{array}{ccc}
K(C\mathbb{P}^{2n-1}) & \xrightarrow{\text{IC}} & K(E^2 \mathbb{CP}^{2n-1}) \\
\downarrow \text{ch} & & \downarrow \text{ch} \\
H^*(C\mathbb{P}^{2n-1}, \mathbb{Q}) & \xrightarrow{\sigma^{-2}} & H^*(E^2 \mathbb{CP}^{2n-1}, \mathbb{Q}).
\end{array}
\]

We denote by \( y \) a generator of \( H^2(C\mathbb{P}^{2n-1}) \). It is also well known that

\[
\text{ch}^{2n-1} y = 1/(2n-1)! y^{2n-1}.
\]

Proposition 3.4. \( \sigma(E^m \omega_{n-1}) \) is a multiple of \( (2n-1)! \) for \( m \geq 0 \).

Proof. The assertion is a direct consequence of Theorem 1.2 of [6] and Proposition 2.6. For the later use, we give another proof for even \( m \).

By (2.4) and Lemma 2.5, \( t_n: H^{4n-1}(E \mathbb{CP}^{2n-1}) \to H^{4n-1}(Q_n(H)) \) is an isomorphism. So, \( y' = t_n^{-1} y^{2n-1} \) is taken as a generator of \( H^{4n-1}(Q_n(H)) \). We choose a generator \( x \) of \( H^{4n-1}(Q_n(H)) \) satisfying \( \langle y', x \rangle = 1 \), where \( \langle , \rangle \) denotes the Kronecker index.

Put \( \sigma(E^m \omega_{n-1}) = k(n) \). Denote by \( s: \tilde{\mathbb{H}}_1(\cdot) \to \tilde{\mathbb{H}}_{i+1}(\mathbb{E}) \) the suspension isomorphism. Then, by Lemma 2.1, there exists an element \( \alpha \in \pi_{m+4n-1}(E^m Q_n(H)) \) satisfying \( \alpha \in k(n)s^m x \). By the definition of the Hurewicz homomorphism, \( h_m(\alpha) = k(n)s^m \xi_n \), where \( \xi_n \) denotes a generator of \( H_{4n-1}(S^{4n-1}) \). So, we have \( k(n) = \langle \sigma^{-m} y', \alpha \rangle = \langle \alpha \sigma^{-m} y', s^m \xi_n \rangle \). Choose a generator \( \tau_n \) of \( H_{4n-1}(S^{4n-1}) \) satisfying \( \langle \tau_n, \xi_n \rangle = 1 \). Then, we have \( \alpha \sigma^{-m} y' = k(n) \sigma^{-m} \tau_n \).

Put \( m = 2t \) and \( u = \text{IC}_n(\mathbb{E}^t) \times \text{IC}_n(v) \in K(E^{m+1}Q_n(H)) \). Then, by (3.2), (3.3) and by the naturality of the Chern character, we have the following:
\( \text{ch}^{2n+1}(E_n) \ast u = \alpha \ast \text{ch}^n_{-1} \ast \alpha^{-1} \text{ch}^{2n-1}(v) = 1/(2n-1)! \alpha \ast \gamma \). 

So, we have \( \text{ch}^{2n+1}(E_n) \ast u = k(n)/(2n-1)! \alpha \ast \gamma \). As is well known, \( \text{Im} \text{ch}^{2n+1} = H^{4n+m}(S^{4n+m}; \mathbb{Z}) \). Hence, \( k(n)/(2n-1)! \) is an integer. This completes the proof.

Lemma 3.5. \((E_n) \ast I_{\nu}(v)\) belongs to the image of the complexification homomorphism \( c': \widetilde{KSp}(EQ_n(H)) \rightarrow \widetilde{K}(EQ_n(H)) \).

Proof. By Lemmas 2.3 and 3.1, \( u' = (E_n) \ast I_{\nu}(v) = (\text{adj} (k \ast i \ast j_2)(C)) \ast (E_n) = (\text{adj} k)_*(E_n) \ast (E_n)(H) \).

Let \( c_\nu: B\nu(n) \rightarrow BU(2n) \) be the mapping induced from \( c: Sp(n) \rightarrow U(2n) \) and \( k': Sp(n) \rightarrow \Omega B\nu(n) \) be the canonical homotopy equivalence. Then, it is well known that \( k \cdot c = c, k' \cdot c \). So, we have \( \text{adj} k)_*(E_n) = (\rho_c)_*(\text{adj} k') \ast \rho_c \ast c \ast c' \). Hence, \( u' = (\rho_c)_*(\text{adj} k') \ast (E_n)(H) \in \text{Im} c' \). This completes the proof.

As is well known, the following diagram commutes:

\[
\begin{array}{ccc}
\widetilde{KSp}(E^8) & \xrightarrow{c'} & \widetilde{K}(E^8) \\
\downarrow_{I^4} & & \downarrow_{I^4} \\
\widetilde{KSp}(E^8) & \xrightarrow{c'} & \widetilde{K}(E^8),
\end{array}
\]

where \( I^4 \) denotes the Bott periodicity isomorphism.

Proposition 3.7. If \( n \) is even and \( m \equiv 0 \mod 8 \), \( \alpha(E_{n-1}^m) \) is a multiple of \( 2 \cdot (2n - 1)! \).

Proof. As is well known, the following diagram commutes:

\[
\begin{array}{ccc}
\widetilde{KSp}(E_{n-1}^m) & \xrightarrow{(E_n) \ast} & \widetilde{KSp}(S^{4n+m}) \\
\downarrow_{c'} & & \downarrow_{c'} \\
\widetilde{K}(EQ_n(H)) & \xrightarrow{(E_n) \ast} & \widetilde{K}(S^{4n+m}),
\end{array}
\]
and \( \text{Im } c = 2K(S^{4n+m}) \) if \( n \) is even. So, by Lemma 3.5, (3.6) and by the proof of Proposition 3.4, \((Ea)\ast u = (Ea)\ast_{C}^{t}(Et_{n})\ast_{C}^{t}(v) \in 2K(S^{4n+m})\) and \(ch^{n+t}(Ea)\ast u \in 2H^{4n+m}(S^{4n+m}; \mathbb{Z})\). Therefore, \( k(n)/(2n-1)! \) is an even integer. This completes the proof.

Remark 2. By the similar arguments, we have the following for \( k \geq 1 \) (cf. [7]):

(1) \( o(E^{k}S_{n-1}(C)) \) is a multiple of \( n! \) for even \( k \).

(2) \( o(E^{k}S_{n-1}(H)) \) is a multiple of \( (2n)!/2 \) for even \( k \). If \( n \) is even and \( k \equiv 0 \mod 8 \), \( o(E^{k}S_{n-1}(H)) \) is a multiple of \( (2n)! \).

§ 4. Proof of the theorem

To prove ii) of our theorem, we use the following [3]:

Theorem 4.1 (James). The stunted quasi-projective space \( Q_{n}(F)/Q_{n-k}(F) \) is an S-retract of the factor space \( G_{n}(F)/G_{n-k}(F) \) for \( k \leq n \). In particular, \( j_{n-k}^{S}: \pi_{i}^{S}(Q_{n}(H)) \longrightarrow \pi_{i}^{S}(Sp(n)) \) is a monomorphism for \( i \geq 0 \).

Now we are ready to prove the theorem. The assertion i) is a direct consequence of Propositions 1.3.1) and 3.7.

By Theorem 4.1, \( j_{n-1}^{S}: \pi_{4n-2}^{S}(Q_{n-1}(H)) \longrightarrow \pi_{4n-2}^{S}(Sp(n-1)) \) is a monomorphism. So, we have \( o(E^{\infty}S_{n-1}) = o(E^{\infty}j_{n-1}^{S}S_{n-1}) \). Therefore, (1.4) and Proposition 3.4 lead us to the assertion. This completes the proof of the theorem.

Remark 3. We can give an improved proof of Theorem 1.2 of [6]. We use the first half of the proof of Theorem 1.2 of [6] and Remark 2.1). We have

(1) \( o(E^{k}S_{n-1}(C)) = n! \) for \( k \geq 1 \).

By (1) and Remark 2.2), we have the following:

(2) If \( n \) is even, \( o(E^{k}S_{n-1}(H)) = (2n)! \) for \( k \geq 1 \).
By Theorem 1.1 of [7] and by Lemma 2.1,

(3). $o(E_{n-1}^{\infty} (H)) = (2n)!/2$ if $n$ is odd.

In this case, the Adams spectral sequence is used for the 2-primary stable homotopy of quaternionic and complex projective spaces [7].

§ 5. An example

An open problem is to determine the order of $\omega_n (H)$ completely. The author hopes that an affirmative answer is given to the following

Conjecture. $o(\omega_{n-1} (H)) = (2n - 1)!$ if $n$ is odd.

In this section, we determine the group structure of $\pi_{10}(Q_2 (H))$ and we show that the conjecture is true for $n = 3$. We use the following: $\pi_{11}(S^3) \cong \mathbb{Z}_2$, $\pi_{10}(S^7) = \{v_7\} \cong \mathbb{Z}_{24}$, $\pi_{11}(S^7) \cong 0$, $\pi_9(S^3) \cong \mathbb{Z}_3$ and $\pi_{10}(S^3) \cong \mathbb{Z}_{15}$.

Example. $\pi_{10}(Q_2 (H)) \cong \mathbb{Z}_{5!} + \mathbb{Z}_2$ and $o(\omega_2 (H)) = 5!$.

Proof. Let $p: (Q_2 (H), S^3) \to (S^7, \ast)$ be the collapsing map. Then, $p_*: \pi_7(Q_2 (H), S^3) \to \pi_7(S^7)$ is an isomorphism [1]. We choose a generator $\alpha$ of $\pi_7(Q_2 (H), S^3) \cong \mathbb{Z}$ such that $p_* \alpha = v_7$.

$Sp(2)$ is regarded as the cell complex $Q_2 (H) \cup e_7, 3$. Let $p': (Sp(2), Q_2 (H)) \to (S^{10}, \ast)$ be the collapsing map. Then, $p'_*: \pi_n(Sp(2), Q_2 (H)) \to \pi_n(S^{10})$ is an isomorphism for $n \leq 11$ [1].

We consider the following commutative diagram:
\[ \pi_{11}(\text{Sp}(2), S^3) \cong_0 \]

\[ \pi_{11}(\text{Sp}(2), \mathcal{Q}_2(H)) = \pi_{11}(\text{Sp}(2), \mathcal{Q}_2(H)) \]

\[ \downarrow \theta \quad \downarrow \theta' \]

\[ \pi_{10}(S^3) \xrightarrow{i_{10}} \pi_{10}(\mathcal{Q}_2(H)) \xrightarrow{i_{10}} \pi_{10}(\mathcal{Q}_2(H), S^3) \xrightarrow{\theta''} \pi_9(S^3) \]

\[ \pi_{11}(S^7) \xrightarrow{\cong_0} \pi_{10}(S^3) \xrightarrow{i_{10}} \pi_{10}(\text{Sp}(2)) \xrightarrow{P_{10}} \pi_{10}(S^7) \xrightarrow{\Delta'} \pi_9(S^3) \xrightarrow{\cong_0} \pi_9(\text{Sp}(2)) \]

where the mappings are canonical and the horizontal and perpendicular sequences are exact respectively.

\[ P_* \] is a split epimorphism since \[ P_*(\lambda \gamma) = \nu_7 \]. So, we have \[ \pi_{10}(\mathcal{Q}_2(H), S^3) \cong \mathbb{Z}_{24} + \mathbb{Z}_2 \]. By the commutativity of the above diagram, \[ i_* \] is a monomorphism and \[ \theta'' \] is an epimorphism. Therefore, by the upper horizontal sequence, \[ \pi_{10}(\mathcal{Q}_2(H)) \cong \mathbb{Z}_{5!} + \mathbb{Z}_2 \]. Hence, by Proposition 1.3.ii), we have \[ o(\omega_2) = 5! \]. This completes the proof.
References


