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Kyoto University
The Order of the Attaching Class of the Suspended Quaternionic Quasi-Projective Space

By

Juno Mukai

§ 0. Introduction

In this note, $F$ denotes the field of the complex numbers $C$ or the field of the quaternions $H$. We denote by $F^n$ the $F$-projective space of $n$ $F$-dimensions and by $Q_n(F)$ the quasi-$F$-projective space. $G_n(F)$ denotes the unitary group $U(n)$ or the symplectic group $Sp(n)$ according as $F$ is $C$ or $H$. Let $d$ be the dimension of $F$ over the field of the real numbers $R$ and $S^{dn-1}$ the unit sphere in $F^n$. Let $T_n: S^{dn-2} \to G_{n-1}(F)$ be the characteristic map for the normal form of the principal $G_{n-1}(F)$-bundle over $S^{dn-1}$. Then, as is well known ( [2], [3] and [9] ), $\text{Im } T_n = Q_{n-1}(F)$, precisely, the following diagram commutes:

$$
\begin{array}{ccc}
S^{dn-2} & \xrightarrow{T_n} & Q_{n-1}(F) \\
\downarrow{T_n'} & & \downarrow{j_{n-1}} \\
G_{n-1}(F), & & \\
\end{array}
$$

where $j_{n-1}$ is the canonical reflection map. $Q_n(F) = Q_{n-1}(F) \cup_{T_n} e^{dn-1}$ and $Q_n(C) = E(CP^{n-1})$, where $E(\ )$ denotes the reduced suspension and $CP^{n-1}$ a disjoint union of $CP^{n-1}$ and $\{\text{one point}\}$.

Let $\omega_{n-1} = \omega_{n-1}(F)$ be the homotopy class of $T_n$ and $p: Q_n(C) \to Q_n(C)/Q_1(C) = ECP^{n-1}$ the collapsing map. In the previous paper [6], we proved that the $k$-th suspension $E^k(p_*\omega_n(C))$ is of order $n!$ for $k \geq 0$.

The purpose of this note is to examine the order of $E^k\omega_{n-1}(H)$. 

- 1 -
Let \( \alpha \) be an element of a homotopy group \( \pi_n(\ ) \) and \( E^\infty_\alpha \in \pi_n(\ ) \) the stable element of \( \alpha \). \( o(\beta) \) denotes the order of \( \beta \). Then, our result is the following

Theorem. i). \( o(E^k_{\omega_{n-1}^n}(H)) = 2^* (2n-1)! \) for \( k \geq 0 \) if \( n \) is even.

ii). \( o(E^\infty_{\omega_{n-1}^n}(H)) = (2n-1)! \) if \( n \) is odd.

Our method is essentially to use the K-theory. To examine \( o(\omega_{n-1}^n(H)) \), we use the Toda's theorem about the generator of \( \pi_{2n-1}(U(n)) \) [6] and the group structure of \( \pi_{4n+2}(Sp(n)) \) [4]. To determine the lower bound of \( o(E^k_{\omega_{n-1}^n}(H)) \), we use the standard method of D. M. Segal [7] from the unstable viewpoint, exactly, we use the Hurwicz homomorphism \( h : \pi_{k+4n-1}(E^k_{Q_n} (H)) \rightarrow H_{k+4n-1}(E^k_{Q_n} (H); \mathbb{Z}) \). A powerful tool is the Toda-Kozima's map \( \tilde{\tau}_n : Q_n^* (H) \rightarrow Q_{2n}^* (C) \) [8].

Our result overlaps partially with the works of K. Morisugi [5] and G. Walker [9].

The author wishes to express his sincere gratitude to Professor S. Sasao for many advices given during the preparation of this paper.

§1. Determination of \( o(\omega_{n-1}^n(H)) \) for even \( n \)

First we recall the definition of the quasi-projective space and the reflection map. \( S(F^n) \) denotes the unit sphere in \( F^n \). \( Q_n(F) \) is the space obtained from \( S(F^n) \times S(F) \) by imposing the equivalence relation: \( (u, q) \sim (ug, g^{-1} qg) \) for \( g \in S(F) \) and collapsing \( S(F^n) \times \{1\} \) to a point. The reflection map \( j_n = j_n(F) : Q_n(F) \rightarrow G_n^F(F) \) is defined as follows:

\[
    j_n([u, q])(v) = v + u(q - 1)\langle u, v \rangle
\]

for \( u \in S(F^n), q \in S(F) \) and \( v \in F^n \), where \( \langle u, v \rangle = \sum_{k=1}^n u_k v_k \) for \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_n) \).
Let $z = x + jy \in \mathbb{H}$, where $x, y \in \mathbb{C}$. By regarding $x \in \mathbb{C}$ as $x + j0 \in \mathbb{H}$, we have the injection $\mathbb{C} \hookrightarrow \mathbb{H}$. Obviously, this induces the canonical maps $i_n^1: Q_n(\mathbb{C}) \rightarrow Q_n(\mathbb{H})$ and $i_n^1: U(n) \rightarrow \text{Sp}(n)$. From the definition, the following diagram commutes:

\[
\begin{array}{ccc}
Q_n(\mathbb{C}) & \xrightarrow{i_n^1} & Q_n(\mathbb{H}) \\
\downarrow j_n & & \downarrow j_n \\
U(n) & \xrightarrow{i_n^1} & \text{Sp}(n).
\end{array}
\]

In the complex case, we can define the reduced reflection map [6]:

$j_n^1 = j_n^1(\mathbb{C}): \mathbb{C}P^{n-1} \cong Q_n(\mathbb{C})/Q_1(\mathbb{C}) \rightarrow U(n)/U(1) \cong SU(n).

By abuse of notation, we often use the same letter $j_n$ for the reduced case.

Lemma 1.2. i). If $n$ is even, $j_n^*: \pi_{4n-1}(Q_n(\mathbb{H})) \rightarrow \pi_{4n-1}(\text{Sp}(n))$ is an epimorphism.

ii). If $n$ is odd, Im $j_n^* = a\pi_{4n-1}(\text{Sp}(n))$, where $a = 1$ or 2.

Proof. Let $p: Q_{2n}(\mathbb{C}) \rightarrow Q_{2n}(\mathbb{C})/Q_1(\mathbb{C}) \cong \mathbb{C}P^{2n-1}$ be the collapsing map, $k: Q_n(\mathbb{H}) \rightarrow Q_{2n}(\mathbb{H})$ and $k': \text{Sp}(n) \rightarrow \text{Sp}(2n)$ the inclusion maps, respectively. Then, by (1.1), the following diagram commutes for $r = 4n - 1$:

\[
\begin{array}{ccc}
\pi_r(\mathbb{C}P^{2n-1}) & \xleftarrow{p_*} & \pi_r(Q_{2n}(\mathbb{C})) \\
\downarrow j_{2n}(\mathbb{C})_* & & \downarrow j_{2n}(\mathbb{C})_* \\
\pi_r(U(2n)) & \xrightarrow{i_{2n}^1} & \pi_r(\text{Sp}(2n)) \\
\downarrow k_* & & \downarrow k_* \\
\pi_r(\text{Sp}(n)) & = & \pi_r(\text{Sp}(n)).
\end{array}
\]

$p_*$ is an epimorphism since $Q_{2n}(\mathbb{C}) \cong \mathbb{C}P^{2n-1}\vee S^1$. By Theorem 4.1 of [6], $j_{2n}(\mathbb{C})_*$ is an epimorphism. So, $j_{2n}(\mathbb{C})_*$ is an epimorphism. $k_*$ and $k'_*$ are isomorphisms respectively. As is well known, $i_{2n}^1$ is an isomorphism if $n$ is even and Im $i_{2n}^1 = 2\pi_{4n-1}(\text{Sp}(2n))$ if $n$ is odd. Therefore, the above commutative diagram leads us to the assertion. This completes the proof.
Proposition 1.3. \(i\). \(\circ(\omega_{n-1}) = 2 \cdot (2n - 1)! \) for even \(n\).
\(\circ(\omega_{n-1}) = 2 \cdot (2n - 1)! \) for odd \(n\), where \(a\) is the
same number as in Lemma 1.2.

Proof. Let \(p: (Q_n(H), Q_{n-1}(H)) \to (S^{4n-1}, \ast)\) be the collapsing map. We
consider the natural homomorphism between the exact sequences for \(r = 4n - 1\):

\[
\begin{array}{ccccccc}
\pi_r(Q_n(H)) & \overset{j_r}{\longrightarrow} & \pi_r(Q_n(H), Q_{n-1}(H)) & \overset{\partial}{\twoheadrightarrow} & \pi_{r-1}(Q_{n-1}(H)) & \longrightarrow & \pi_{r-1}(Q_n(H)) \\
\downarrow j_n^* & & \downarrow p_* & & \downarrow j_{n-1}^* & & \downarrow j_n^* \\
\pi_r(Sp(n)) & \overset{p_*}{\longrightarrow} & \pi_r(S^{4n-1}) & \overset{\Delta'}{\longrightarrow} & \pi_{r-1}(Sp(n-1)) & \longrightarrow & \pi_{r-1}(Sp(n))
\end{array}
\]

where the mappings are canonical and \(\partial\) and \(\Delta'\) are the connecting homomorphisms.

As is well known, \(\pi_{4n-1}(Sp(n)) \approx \mathbb{Z}\), \(\pi_{4n-2}(Sp(n)) \approx 0\) and \(\pi_m(S^m) = \{1_m\} \approx \mathbb{Z}\). By the Blakers-Massey theorem [1], \(p_*\) is an isomorphism. By the definition,
\(\omega_{n-1} = \Delta(\omega_{4n-1})\), where \(\Delta = \partial \cdot p_*^{-1}\). So, by Theorem 2.2 of [4], \(j_{n-1}^*\) is an epimorphism and the following holds:

\[(1.4) \quad \pi_{4n-2}(Sp(n-1)) = \{j_{n-1}^* \circ \omega_{n-1} \} \approx \mathbb{Z} \cdot b \cdot (2n-1)!, \quad \text{where } b = 1 \text{ for odd } n \text{ and } b = 2 \text{ for even } n.\]

By the exactness of the upper sequence, \(\circ(\omega_{n-1})\) is equal to the order of the
co-kernel of \(j_n^*\). Hence, by (1.4), Lemma 1.2 and by the above commutative dia-
gram, we have the assertion. This completes the proof.

By inspecting the above proof, we have the following

Proposition 1.5. \(j_n^*: \pi_{4n-1}(Q_n(H)) \to \pi_{4n-1}(Sp(n))\) is an epimorphism if
and only if \(\circ(\omega_{n-1}) = b \cdot (2n - 1)!\), where \(b\) is the same number as in (1.4).
2. Some fundamental facts

For \( n \geq 0 \), \( X_n \) denotes a connected finite CW complex such that \( X_0 = \{ \ast \} \) and \( X_n = e^0 \cup e^1 \cup \ldots \cup e^r \) for \( n \geq 1 \). Here \( r = r_n = dn - \varepsilon \) with \( \varepsilon = 0 \) or 1 and \( d - \varepsilon \geq 2 \). \( \theta_{n-1} : S^{r-1} \to X_{n-1} \) denotes the attaching map, and so \( X_n = X_{n-1} \cup_{\theta_{n-1}} e^r \). For example, \( X_n = FP^n \) (\( d = 2 \) or 4 and \( \varepsilon = 0 \)) and \( X_n = Q_n(H) \) (\( d = 4 \) and \( \varepsilon = 1 \)).

Let \( p : X_n \to X_n/X_{n-1} = S^r \) and \( p' : (X_n, X_{n-1}) \to (S^r, \ast) \) be the collapsing maps. Let \( \beta : \pi_{r+m}(E^mX_n, E^mX_{n-1}) \to \pi_{r+m-1}(E^mX_{n-1}) \) be the connecting homomorphism. Then, \((E^m \delta)^* : \pi_{r+m}(E^mX_n, E^mX_{n-1}) \to \pi_{r+m}(S^{r+m}) \) is an isomorphism for \( m \geq 0 \) [1], and we define a homomorphism \( \Delta : \pi_{r+m}(S^{r+m}) \to \pi_{r+m-1}(E^mX_{n-1}) \) by the composition \( \beta \circ (E^m \delta)^{-1} \). By the definition, \( \Delta(\iota_{r+m}) = E^m \delta_{n-1} \), where the same letter is used for a mapping and its homotopy class.

Let \( h = h : \pi_{r+m}(E^mX_n) \to H_{r+m}(E^mX_n; \mathbb{Z}) \approx \mathbb{Z} \) for \( m \geq 0 \) be the Hurewicz homomorphism and \( h(n, m) \) the non-negative integer such that \( \text{Im} \ h = h(n, m) \).

Lemma 2.1. \( o(E^m \delta_{n-1}) = h(n, m) \).

Proof. \( j : (E^mX_n, \ast) \to (E^mX_n, E^mX_{n-1}) \) denotes the inclusion. Then, we consider the commutative diagram:

\[
\begin{array}{ccc}
\pi_{r+m}(E^mX_n) & \xrightarrow{j_*} & \pi_{r+m}(E^mX_n, E^mX_{n-1}) \\
\downarrow h & & \downarrow h' \\
H_{r+m}(E^mX_n; \mathbb{Z}) & \xrightarrow{j_*} & H_{r+m}(E^mX_n, E^mX_{n-1}; \mathbb{Z}),
\end{array}
\]

where \( h' \) denotes the relative Hurewicz homomorphism and the upper sequence is exact. From the cell structure of \( X_n \), the lower \( j_* \) is an isomorphism. By the relative Hurewicz theorem, \( h' \) is an isomorphism. This completes the proof.
According to [8], a representative element of $Q_n(H)$ can be taken as
$(x + jy, e^{i\pi t})$, where $x, y \in \mathbb{C}^n$ satisfying $x + jy \in S(H^n)$ and $0 \leq t \leq 1$. Toda
and Kozima defined $\xi_n: Q_n(H) \rightarrow Q_{2n}(\mathbb{C})$ by the equation
$$\xi_n[(x + jy, e^{i\pi t})] = [(x \oplus y, e^{2i\pi t})].$$

We define $t_n: Q_n(H) \rightarrow ECP^{2n-1}$ by the composition $\pi \cdot \xi_n$, where $p: Q_{2n}(\mathbb{C})$
$\rightarrow ECP^{2n-1}$ is the collapsing map. From the definition, the following
diagram commutes for $k < n$:

\[
\begin{array}{ccc}
Q_k(H) & \xrightarrow{t_k} & ECP^{2k-1} \\
\downarrow i & & \downarrow i' \\
Q_n(H) & \xrightarrow{t_n} & ECP^{2n-1},
\end{array}
\]

where $i$ and $i'$ the canonical inclusions.

The following lemma is a reduced version of Proposition 2.5 of [8].

**Lemma 2.3 (Toda-Kozima).** The following diagram commutes up to
homotopy:

\[
\begin{array}{ccc}
 Q_n(H) & \xrightarrow{t_n} & ECP^{2n-1} \\
\downarrow j_n & & \downarrow j_{2n} \\
 Sp(n) & \xrightarrow{c} & SU(2n),
\end{array}
\]

where $c$ is the complexification map.

Let $p: Q_n(H) \rightarrow Q_n(H)/Q_{n-1}(H) = S^{4n-1}$ for $n \geq 1$ and $p': ECP^{2n-1} \rightarrow$
$ECP^{2n-1}/ECP^{2n-3} = S^{4n-3} \vee S^{4n-1}$ for $n \geq 2$ be the collapsing maps. Then,
by (2.2), there exists a mapping $t'_n: S^{4n-1} \to S^{4n-3} \lor S^{4n-1}$ for $n \geq 2$ such that the following diagram commutes:

$$
\begin{array}{ccc}
Q_n(H) & \xrightarrow{t_n} & ECP^{2n-1} \\
\downarrow p & & \downarrow p' \\
S^{4n-1} & \xrightarrow{t'_n} & S^{4n-3} \lor S^{4n-1}.
\end{array}
$$

(2.4)

Let $p_2: S^{4n-3} \lor S^{4n-1} \to S^{4n-1}$ for $n \geq 2$ be the projection map. Then, we have the following

**Lemma 2.5.** deg $t_1 = -1$ and deg $(p_2t'_n) = (-1)^n$ for $n \geq 2$.

**Proof.** We define $q_n: S(H^n) \to S(C^{2n})$ by the equation

$$q_n(x + jy) = x \oplus y$$

for $x, y \in C^n$. It is clear that $q_n$ is a homeomorphism and deg $q_n = (-1)^n$.

By Lemma 2.3, $t_1 = q_1$ and $p_2t'_n = q_n$ for $n \geq 2$. This completes the proof.

Hereafter the same letter is often used for a mapping and its homotopy class. Let $\gamma_n = \gamma_n(F): S(F^{n+1}) \to Fr^n$ be the projection map. Let $i: ECP^{2n-1} \to ECP^{2n}$ be the inclusion map. Then, we have the following

**Proposition 2.6.** $(-1)^{n+1} \Sigma_2\gamma_n(C) = i \omega_n(H)$.

**Proof.** By (2.2) and (2.4), the following diagram commutes for $r = 4n + 3$: 
\[ \pi_r(S^{4n+3}) \xrightarrow{p_*} \pi_r(Q_{n+1}(H), Q_n(H)) \xrightarrow{\beta} \pi_{r-1}(Q_n(H)) \]

\[ \downarrow t_{n+1}^* \quad \downarrow t_{n+1}^* \quad \downarrow t_n^* \]

\[ \pi_r(S^{4n+1} \cup S^{4n+3}) \xrightarrow{p'_*} \pi_r(\text{ECP}^{2n+1}, \text{ECP}^{2n-1}) \xrightarrow{\beta'} \pi_{r-1}(\text{ECP}^{2n-1}) \]

\[ \downarrow p_{2*} \quad \downarrow i_* \quad \downarrow i_* \]

\[ \pi_r(S^{4n+3}) \xrightarrow{p''_*} \pi_r(\text{ECP}^{2n+1}, \text{ECP}^{2n}) \xrightarrow{\beta''} \pi_{r-1}(\text{ECP}^{2n}) \]

where the mappings are canonical.

\[ p_* \text{ and } p''_* \text{ are isomorphisms} \]

We note that \( \omega_n(H) = \partial p_{n-1}^{-1}(\lambda_{4n+3}) \) and \( EY_{2n}(C) = \partial'' p_{n-1}^{-1}(\lambda_{4n+3}) \). So, by Lemma 2.5 and the above commutative diagram, we have the assertion. This completes the proof.

Remark 1. Owing to Proposition 2.6, it suffices to take \((-1)^{n+1} t_{\omega_n} \lambda_{2n} \) as \( \lambda_n \) in Proposition 6.5.ii) of [6]. By Theorem 1.2 of [6] and Proposition 1.3, \( o(\lambda_{2n}) = (2n + 1)! \) or \( 2 \cdot (2n + 1)! \). In the last section, we shall show that \( o(\lambda_4) = 5! \) (cf. Lemma 11.1 of [6]).

§ 3. Determination of the lower bound of \( o(E^m_{n-1}(H)) \).

Let \( v \in R(\text{CP}^{2n-1}) \) be the stable isomorphism class of the canonical line bundle over \( \text{CP}^{2n-1} \). We denote by \( I_C : R() \to R(E^2) \) the Bott periodicity isomorphism. The following Lemma is well known (cf. Lemma 2.2 of [8]).

Lemma 3.1. \( I_C(v) \in R(E^2 \text{CP}^{2n-1}) \) is represented by the adjoint of the composite of the canonical maps:

\[ \text{ECP}^{2n-1} \xrightarrow{j_{2n}} \text{SU}(2n) \xrightarrow{i} \text{U}(2n) \xrightarrow{k} \Omega \text{BU}(2n), \]

where \( k \) is the homotopy equivalence.
Hereafter, \( \mathbb{Z} \) or the rational number field \( \mathbb{Q} \) is taken as the coefficients of the homology or cohomology groups, unless otherwise stated.

Let \( \text{ch}^k : K(\cdot) \to H^{2k}(\cdot; \mathbb{Q}) \) be the \( k \)-th Chern character and \( \text{ch} = \sum_k \text{ch}^k \) the total Chern character. Let \( \sigma : \mathbb{R}^i(\cdot) \to R^{i-1}(\cdot) \) be the suspension isomorphism. Then, as is well known, the following diagram commutes:

\[
\begin{array}{ccc}
K(\mathbb{CP}^{2n-1}) & \xrightarrow{\text{IC}} & K(\mathbb{C}P^{2n-1}) \\
\downarrow \text{ch} & & \downarrow \text{ch} \\
H^*(\mathbb{CP}^{2n-1}; \mathbb{Q}) & \xrightarrow{\sigma^2} & H^*(\mathbb{C}P^{2n-1}; \mathbb{Q}).
\end{array}
\]

We denote by \( y \) a generator of \( H^2(\mathbb{CP}^{2n-1}) \). It is also well known that

\[
(3.3) \quad \text{ch}^{2n-1} y = 1/(2n-1)! y^{2n-1}.
\]

Proposition 3.4. \( o(E^m_{n-1}) \) is a multiple of \( (2n-1)! \) for \( m \geq 0 \).

Proof. The assertion is a direct consequence of Theorem 1.2 of [6] and Proposition 2.6. For the later use, we give another proof for even \( m \).

By (2.4) and Lemma 2.5, \( t^*: H^{4n-1}(\mathbb{BC}P^{2n-1}) \to H^{4n-1}(\mathbb{Q}_n(\mathbb{H})) \) is an isomorphism. So, \( y' = t^* \sigma_n^{-1} y^{2n-1} \) is taken as a generator of \( H^{4n-1}(\mathbb{Q}_n(\mathbb{H})) \). We choose a generator \( x \) of \( H_{4n-1}(\mathbb{Q}_n(\mathbb{H})) \) satisfying \( \langle y', x \rangle = 1 \), where \( \langle , \rangle \) denotes the Kronecker index.

Put \( o(E^m_{n-1}) = k(n) \). Denote by \( s : R_i(\cdot) \to R_{i+1}(\mathbb{E}) \) the suspension isomorphism. Then, by Lemma 2.1, there exists an element \( \alpha \in \pi_{m+4n-1}(E^m_{\mathbb{Q}_n}(\mathbb{H})) \) satisfying \( h_m(\alpha) = k(n)s^m x \). By the definition of the Hurewicz homomorphism, \( m(\alpha) = \alpha^m s^m \xi_n \), where \( \xi_n \) denotes a generator of \( H_{4n-1}(S^{4n-1}) \). So, we have \( k(n) = \langle \sigma^m y', \alpha^m s^m \xi_n \rangle = \langle \alpha \sigma^m y', s^m \xi_n \rangle \). Choose a generator \( \tau_n \) of \( H^{4n-1}(S^{4n-1}) \) satisfying \( \langle \tau_n, \xi_n \rangle = 1 \). Then, we have \( \alpha \sigma^m y' = k(n) \sigma^m \tau_n \).

Put \( m = 2t \) and \( u = I_{C_n}^*(\mathbb{E}^t) \in \mathbb{R}(E^{m+1}_{\mathbb{Q}_n}(\mathbb{H})) \). Then, by (3.2), (3.3) and by the naturality of the Chern character, we have the following:
\[ \sigma_{2n+1}(E) \cdot u = \alpha \cdot \sigma_{-m} \cdot \tau_{n}^{-1} \cdot \sigma_{-n}^{-1} \cdot \chi_{2n+1}(v) = 1/(2n-1)! \alpha \cdot \sigma_{-m} \cdot v'. \]

So, we have \( \chi_{2n+1}(E) \cdot u = k(n)/(2n-1)! \alpha \cdot \sigma_{-m} \cdot \tau_{n}^{-1} \cdot \sigma_{-n}^{-1} \).

As is well known, \( \text{Im } \chi_{2n+1} = H^{4n+m}(S^{4n+m}; \mathbb{Z}) \). Hence, \( k(n)/(2n-1)! \) is an integer. This completes the proof.

**Lemma 3.5.** \((E_n^*)^*_{\mathcal{C}}(v)\) belongs to the image of the complexification homomorphism \(c': KSp(E\mathcal{Q}_n(H)) \rightarrow K(E\mathcal{Q}_n(H))\).

**Proof.** By Lemmas 2.3 and 3.1, \(u' = (E_n^*)^*_{\mathcal{C}}(v) = (\text{adj } (k \cdot i \cdot j_{2n}(\mathcal{C})))^*(E_n^*) = (\text{adj } k)^*(E_n^*)^*(E_j^*)(H)\).

Let \(\rho_c: BSp(n) \rightarrow BU(2n)\) be the mapping induced from \(c: Sp(n) \rightarrow U(2n)\)
and \(k': Sp(n) \rightarrow BBSp(n)\) be the canonical homotopy equivalence. Then, it is well known that \(k \cdot c = \Omega C \cdot k'.\) So, we have \((\text{adj } k)^*(E_n^*) = (\rho_c)^*(\text{adj } k')^*\).

Hence, \(u' = (\rho_c)^*(\text{adj } k')^*(E_j^*)(H) \in \text{Im } c'.\) This completes the proof.

As is well known, the following diagram commutes:

\[
\begin{array}{ccc}
KSp(\cdot) & \xrightarrow{c'} & K(\cdot) \\
\downarrow I_H & & \downarrow I^4 \\
\widetilde{KSp}(\mathcal{E}^8) & \xrightarrow{c'} & K(\mathcal{E}^8),
\end{array}
\]

where \(I_H\) denotes the Bott periodicity isomorphism.

**Proposition 3.7.** If \(n\) is even and \(m \equiv 0 \mod 8\), \(o(E_n^m)\) is a multiple of \(2 \cdot (2n - 1)!\).

**Proof.** As is well known, the following diagram commutes:

\[
\begin{array}{ccc}
\widetilde{KSp}(E^m_{\mathcal{Q}_n}(H)) & \xrightarrow{(Eu)^*} & \widetilde{KSp}(S^{4n+m}) \\
\downarrow c' & & \downarrow c \\
K(E\mathcal{Q}_n(H)) & \xrightarrow{(Eu)^*} & K(S^{4n+m}),
\end{array}
\]

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and \( \text{Im } c = 2^k(S^{4n+m}) \) if \( n \) is even. So, by Lemma 3.5, (3.6) and by the proof of Proposition 3.4, \((Ea)^*u = (Ea)^*t(C^t_n) * t(C)(v) \in 2R(S^{4n+m}) \) and \( ch^{2n+t}(Ea) * u \in 2H^{4n+m}(S^{4n+m}; \mathbb{Z}) \). Therefore, \( k(n)/(2n-1) \) is an even integer. This completes the proof.

Remark 2. By the similar arguments, we have the following for \( k \geq 1 \) (cf. [7]):

1. \( o(E_{\gamma_{n-1}}(C)) \) is a multiple of \( n! \) for even \( k \).
2. \( o(E_{\gamma_{n-1}}(H)) \) is a multiple of \( (2n)!/2 \) for even \( k \). If \( n \) is even and \( k \equiv 0 \mod 8 \), \( o(E_{\gamma_{n-1}}(H)) \) is a multiple of \( (2n)! \).

§ 4. Proof of the theorem

To prove ii) of our theorem, we use the following [3]:

Theorem 4.1 (James). The stunted quasi-projective space \( Q_n(F)/Q_{n-k}(F) \) is an S-retract of the factor space \( G_n(F)/G_{n-k}(F) \) for \( k \leq n \). In particular, \( j_n^*: \pi_1^S(Q_n(H)) \to \pi_1^S(\text{Sp}(n)) \) is a monomorphism for \( i \geq 0 \).

Now we are ready to prove the theorem. The assertion i) is a direct consequence of Propositions 1.3.i) and 3.7.

By Theorem 4.1, \( j_n^{F*}: \pi_{4n-2}(Q_{n-1}(H)) \to \pi_{4n-2}(\text{Sp}(n-1)) \) is a monomorphism. So, we have \( o(E_{\omega_{n-1}}) = o(E_{\gamma_{n-1}}(H)) \). Therefore, (1.4) and Proposition 3.4 lead us to the assertion. This completes the proof of the theorem.

Remark 3. We can give an improved proof of Theorem 1.2 of [6]. We use the first half of the proof of Theorem 1.2 of [6] and Remark 2.(1). We have

1. \( o(E_{\gamma_{n-1}}(C)) = n! \) for \( k \geq 1 \).

By (1) and Remark 2.(2), we have the following:

2. If \( n \) is even, \( o(E_{\gamma_{n-1}}(H)) = (2n)! \) for \( k \geq 1 \).
By Theorem 1.1 of [7] and by Lemma 2.1,

\[ o(E_{\infty}Y_{n-1}(H)) = (2n)!/2 \text{ if } n \text{ is odd.} \]

In this case, the Adams spectral sequence is used for the 2-primary stable homotopy of quaternionic and complex projective spaces [7].

§ 5. An example

An open problem is to determine the order of \( \omega_n(H) \) completely. The author hopes that an affirmative answer is given to the following

Conjecture. \[ o(\omega_{n-1}(H)) = (2n - 1)! \text{ if } n \text{ is odd.} \]

In this section, we determine the group structure of \( \pi_{10}(Q_2(H)) \) and we show that the conjecture is true for \( n = 3 \). We use the following: \( \pi_{11}(S^3) \cong Z_2 \), \( \pi_{10}(S^7) = \langle v_7 \rangle \cong Z_{24} \), \( \pi_{11}(S^7) \cong 0 \), \( \pi_{9}(S^3) \cong Z_3 \) and \( \pi_{10}(S^3) \cong Z_{15} \).

Example. \( \pi_{10}(Q_2(H)) \cong Z_{5!} + Z_2 \) and \( o(\omega_2(H)) = 5! \).

Proof. Let \( p: (Q_2(H), S^3) \to (S^7, *) \) be the collapsing map. Then, \( p_*: \pi_7(Q_2(H), S^3) \to \pi_7(S^7) \) is an isomorphism [1]. We choose a generator \( \alpha \) of \( \pi_7(Q_2(H), S^3) \cong Z \) such that \( p_*\alpha = v_7 \).

\( \text{Sp}(2) \) is regarded as the cell complex \( Q_2(H) \cup e^{7,3} \). Let \( p': (\text{Sp}(2), Q_2(H)) \to (S^{10}, *) \) be the collapsing map. Then, \( p'_*: \pi_n(\text{Sp}(2), Q_2(H)) \to \pi_n(S^{10}) \) is an isomorphism for \( n \leq 11 \) [1].

We consider the following commutative diagram:
\[ \pi_{11}(\text{Sp}(2), S^3) \cong 0 \]

\[ \pi_{11}(\text{Sp}(2), Q_2(H)) = \pi_{11}(\text{Sp}(2), Q_2(H)) \]

\[ \downarrow \quad \text{Z}_2 \]

\[ \pi_{10}(S^3) \xrightarrow{i} \pi_{10}(Q_2(H)) \xrightarrow{j} \pi_{10}(Q_2(H), S^3) \xrightarrow{\alpha''} \pi_9(S^3) \]

\[ \| \quad \| \quad \text{Z}_3 \]

\[ \pi_{11}(S^7) \xrightarrow{\gamma} \pi_{10}(S^3) \xrightarrow{i} \pi_{10}(\text{Sp}(2)) \xrightarrow{F} \pi_{10}(S^7) \xrightarrow{\Delta'} \pi_9(S^3) \rightarrow \pi_9(\text{Sp}(2)), \]

\[ \cong \quad \text{Z}_2 \]

\[ \text{Z}_5! \quad \text{Z}_5! \]

\[ \text{Z} \quad \text{Z} \]

\[ \pi_{10}(\text{Sp}(2), Q_2(H)) \]

where the mappings are canonical and the horizontal and perpendicular sequences are exact respectively.

\[ p_* \] is a split epimorphism since \( p_*(\alpha v) = v \). So, we have \( \pi_{10}(Q_2(H), S^3) \cong \text{Z}_2 \). By the commutativity of the above diagram, \( i_* \) is a monomorphism and \( \alpha'' \) is an epimorphism. Therefore, by the upper horizontal sequence, \( \pi_{10}(Q_2(H)) \cong \text{Z}_5! + \text{Z}_2 \). Hence, by Proposition 1.3.ii, we have \( \sigma(\omega_2) = 5! \). This completes the proof.
References


