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On Some Predictability in 1-dimensional Chaotic Dynamical Systems.

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§1 Introduction.

Vandermeer suggested that some chaotic dynamical systems have a kind of predictable behaviors in their orbits, and he called such systems "resolved chaos". We will give some quantitative discussion for this idea taking account of the fluctuation from deterministic dynamical systems. Vandermeer's "resolved chaos" ha had a background in population biology which describes (n+1)th year's population from n-th year's, and we will consider iteration of 1-dimensional unimodal maps. (See Vandermeer[1].)

A unimodal map is a map with only one local maximum. See §2 for a precise definition. We denote the critical point of a unimodal map f by k. Then, let $x \geqq k$, considering how many times the orbit of x by iterations of f (i.e. $f(x), f^2(x), \cdots\cdots$) stays in $[0,k)$, we denote this by $n(f,x)$ and call it sojourning time of x in $[0,k)$. (See Fig.1. For example the sojourning time of x_0 in Fig.1 is 2.)

In population biology under constant supply of resource the population flush at a year leads a serious struggle for existence, and in consequence next year's population is very rare. Here the sojourning time says how mamy years it will take for the populatio to recover from rarity and return to the level of k.

By the way, if we think the dynamical system given by iterations of a map f as a model of some biological or physical system , it seems natural that the real systems are described by adding some fluctuation to the model, for the procedure of modelling always include a kind of approximation and idealization. Hence we define some kind of neighborhood of f called "δ-neighborhood" and denote it by $U(f,\delta)$ in §2. Here δ represents strength of fluctuation. We will consider random iteration from $U(f,\delta)$, which means that (n+1)th state is given by $x_{n+1} = g_n(x_n)$ where x_n is n-th state and g_n is randomly chosen from $U(f,\delta)$. We can also define sojourning time for such random iteration system and denote it by $n(\{g_i\}_{i=0}^{+\infty},x)$. In general, sojourning time of the modelling systems $n(f,x)$ is easily computed but that of the fluctuating systems has wide range. Therefore we are interested in the following problem.

Problem.

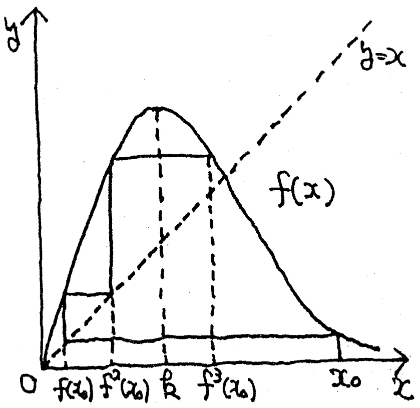
How accurate can we predict sojourning time of fluctuating systems by sojourning time of the modelling systems? And so, is there some relationship between error of the prediction and strength of fluctuation?

Here the error of prediction in the problem is

$$\Delta n(f,\delta) = \sup_{\{g_i\},\{g_i'\}\subset U(f,\delta) \text{ and } x\in[k,f(k)+\delta]} |n(\{g_i\},x)-n(\{g_i'\},x)|$$

In our framework Vandermeer's result is that there exist chaotic unimodal maps which are predictable on sojourning time.

We show that it is reasonable to think entropy as degree of chaos in §3 and a relation between entropy and sojourning time is also given there. For an explicit discussion we give a definition of 1-parameter family "with resolved chaos" in §4. Roughly saying we say that a family is "with resolved chaos" if the system is the more chaotic the more predictable on sojourning time as the parameter of the family tends to infinity. We will study some sufficient condition for the family with resolved chaos in §4. Theorems given there indicate that resolvability of a family depends on the order of increase of entropy of the family. We can apply the results given in §4 to a well known population model by R. May[3]. (See §5 for the detail.)



§2 Notations and Definitions.

Some frequently used notations are collected in this section.

(1) $\mathbb{R}_+ = [0,+\infty)$

(2) ent(f) = topological entropy of a map f.

(3) $\mathrm{ent}_d(f)$ = metric entropy of a map f.

(4) '$f\uparrow[a,b]$' (resp. '$f\downarrow[a,b]$') means that f is strictly monotone increasing (resp. decreasing) on $[a,b]$.

We always consider continuous maps from $\mathbb{R}_+$ to itself except for §3.

Definition 1. "unimodal map"

Let f be a continuous map, $f(0)=0$ and satisfy one of two following conditions.

(A) There exists a positive constant k such that $f\uparrow[0,k]$ and $f\downarrow[k,+\infty)$.

(B) There exist positive constants k and k' such that $f\uparrow[0,k]$, $f\downarrow[k,k']$ and $f\equiv 0$ on $[k',+\infty)$.

Then f is called a unimodal map, and k is called a turning point of f.

To construct a fluctuating system for a given map f, we need to define a set of maps near f called "δ-neighborhood".

Definition 2. "δ-neighborhood"

Let f be continuous and bounded, and δ is an arbitrary positive constant, then $U(f,\delta) = \{g\colon g:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$(1-\delta/\sup(f))\cdot f(x) \leqq g(x) \leqq (1+\delta/\sup(f))\cdot f(x) \quad \text{for all } x\in[0,\sup(f)]\ ,$$

$$0 \leqq g(x) \leqq \sup(f)+\delta \quad \text{for all } x\in[\sup(f)+\delta,+\infty).\ \}$$

is called δ-neighborhood of f.

Letting $\{g_i\}_{i=0}^{+\infty}\subset U(f,\delta)$ and $x_{n+1}= g_n(x_n)$ be taken for a fluctuating system to a deterministic system $x_{n+1}= f(x_n)$, the problem is how we will be able to predict a behavior of the fluctuating system by the non-fluctuating system. Let us begin with our definition of "sojourning time".

Definition 3. "sojourning time"

Let f be a unimodal map and $\{g_i\}_{i=0}^{+\infty}\subset U(f,\delta)$. We denote $\{g_i\}_{i=0}^{+\infty}$ by G. We define sojourning time in $[0,k]$ for G denoted by $n(G,x)$ as $n(G,x) = \inf\{\ m\colon g_m\circ g_{m-1}\circ\cdots\circ g_0(x) \leqq k\ \}$

here if $\{ m: g_m \circ g_{m-1} \cdots \circ g_0(x) \leqq k \} = \phi$, we set $n(G,x) = +\infty$. Especially sojourning time by a map g (i.e. $g_i = g$ for all i) is denoted by $n(g,x)$.

The "length of rarity time per population flush x" which Vandermeer studied equals $n(f,x)-1$. Here, we are interested to know how far sojourning time will be fluctuated as the choice of G from $U(f,\delta)$ varies arbitrarily. After the first iteration of $g_0 \in U(f,\delta)$, the value $g_0(x)$ must be smaller than $f(k)+\delta$. Hence we are permitted to estimate sojourning time for $x \in [k, f(k)+\delta]$.

Definition 4. estimate of sojourning time

We define $\Delta n(f,\delta)$ such as

$$\Delta n(f,\delta) = \begin{cases} 0 & \text{if } f(k)+\delta < k \quad \text{otherwise} \\ \sup\limits_{x \in [k,f(k)+\delta] \text{ and } G,G' \subset U(f,\delta)} |n(G,x)-n(G',x)| \end{cases}$$

where if both $n(G,x)$ and $n(G',x)$ equal $+\infty$, we set

$$n(G,x)-n(G',x) = 0.$$

To take primarily the supremum of the definition for G and G', we have a following lemma.

Lemma 2-1.

If $k \leqq f(k)+\delta$, then

$$\Delta n(f,\delta) = \sup_{x \in [k,f(k)+\delta]} |n(f_+,x)-n(f_-,x)|$$

where $f_\pm(x) = (1 \pm \delta/f(k))f(x)$ respectively.

Through "toughness" defined right after we know how large we can take δ if we want to predict the sojourning time of a fluctuating system from non-fluctuating one's with a error of ± 1.

Definition 5. "toughness"

$T(f) = \sup\{\delta: n(f,\delta) \leqq 1\}$ is called a toughness of f.

§3 Degree of chaos and sojourning time.

First we consider a unimodal map from a closed interval to itself.

Definition 6.

Let I be a closed interval and $I=[\alpha,\beta]$. Then a map f from I to itself is called unimodal if there exists k such that $\alpha<k<\beta$, $f\uparrow[\alpha,k]$ and $f\downarrow[k,\beta]$.

Here we also define sojourning time just same as Definition 3 Let f be a unimodal map from a closed interval I to itself, $f(k)\geqq k$ and define $N(f)$ by $N(f)=n(f,f(k))$. Then we can give a relation between $N(f)$ and $\mathrm{ent}(f)$ by Milnor-Thurston's kneading theory. (See Milnor-Thurston[2].)

Theorem 3-1.

Let f be a unimodal map from a closed interval I to itself with $f(k)\geqq k$. Let w_n be 1/(positive minimal zero of $t^{n+1}-2t+1$). Then we have

(i) $w_n<w_{n+1}$ and $w_n\uparrow 2$ as $n\uparrow+\infty$

(ii) If $N(f)<+\infty$, then $\log(w_{N(f)}) \leqq \mathrm{ent}(f) \leqq \log(w_{N(f)+1})$,
otherwise $\mathrm{ent}(f)=2$.

It is well known that $\mathrm{ent}(f)>0$ leads the chaos studied by Li-Yorke[4] in the interval dynamical system given by iterations of f. In case of the chaos studied by Li-Yorke, we often observe the so called "window phenomena", that is there exist stable periodic points and almost every points of the interval are attracted to the stable periodic points. However we need to observe the system for infinitly long time if we want to see that an orbit becomes asymptotically periodic. In the present paper we discuss a "transient" orbit starting from $x \geqq k$ and we don't discuss whether the orbit will be attracted to some periodic point or not. Complexity of a "transient" orbit is represented by topological entropy if we recall the definition of it and therefore we can regard entropy as a degree of chaos here.

Through Theorem 3-1 we see that $N(f)$ is a characteristic number for the degree of chaos of a unimodal map f.

Now, let us return to a unimodal map from $\mathbb{R}_+$ to itself. If $f(k) \geqq k$, f can be confined on $[0,f(k)]$ and the chaotic behavior of iterations of f take place in $[0,f(k)]$. Therefore we may consider a interval map $f|_{[0,f(k)]}$ in place of f.

Lemma 3-2

Let f be a unimodal map from $\mathbb{R}_+$ to itself and $f(k) \geqq k$. Then we have $\mathrm{ent}_d(f) = \mathrm{ent}(f|_{[0,f(k)]})$.

This lemma gives a corollary of Theorem 3-1.

Corollary 3-3

Let f be a unimodal map from $\mathbb{R}_+$ to itself, $f(k) \geqq k$ and define $N(f) = n(f,f(k))$. Then we have

$$\log(w_{N(f)}) \leqq \mathrm{ent}_d(f) \leqq \log(w_{N(f)+1}) \quad \text{if} \quad N(f) < +\infty,$$

otherwise $\mathrm{ent}_d(f)=2$.

Hence by Corollary 3-3 $N(f)$ is also a characteristic number for the degree of chaos in case of a unimodal map from $\mathbb{R}_+$ to itself.

We give a sketch of the proof of Theorem 3-1.

Step 1. If $N(f)=0$, then $ent(f)=0$.

(We can easily see that the kneading determinant of f is $1/1-t$ r $1/1+t$.)

Step 2. If $N(f)\geqq 2$, then $ent(f)$ is positive.

(There exists 3-periodic point of f.)

Step 3. Suppose $ent(f)$ be positive, and let $s = \exp(ent(f))$. Then there exists a continuous nontrivial nondecreasing map g from I to $[0,2]$ such that following diagram commutes.

$$\begin{array}{ccc} I & \xrightarrow{g} & [0,2] \\ f\downarrow & & \downarrow F_s \\ I & \xrightarrow[g]{} & [0,2] \end{array} \qquad \text{Here, } F_s(x) = \begin{cases} s\cdot x & \text{on } [0,1] \\ s\cdot(2-x) & \text{on } [1,2]. \end{cases}$$

(This is a direct consequence of Theorem 7-4 of Milnor-Thurston[2].)

Step 4. Under the situation of Step 3, $N(f)=N(F_s)$ or $ent(f)=\log(w_{N(f)})$.

Step 5. We can verify that Theorem 3-1 is true for the piecewise linear map F_s by caluculating directly.

Step 6. Now we are ready to complete the proof of Theorem 3-1. We may consider only these four cases by Step 1 and Step 2.

$$\begin{cases} N(f)=0, \text{ then it follows by Step 1.} \\ N(f)=1 \text{ and } ent(f)=0, \text{ then it follows by the fact that } w_1=0. \\ N(f)<+\infty \text{ and } ent(f)>0, \text{ then it follows by Step 3,4 and 5.} \\ N(f)=+\infty, \text{ then directly caluculation shows that the kneading} \end{cases}$$

determinant is $(1-2t)/(1-t)$. Therefore $ent(f)$ is $\log 2$. //

§4 Prediction of sojourning time and resolved chaos.

Now using the conceptions defioned in the previous sections, we can st te a little quantitatively Vandermeer"s "resolved chaos" such as "If a map with high degree of chaos has large toughness, then we say that the map is with resolved chaos.". This statement is not enough for mathematical discussion because the words "high" and "large" in it are very vague in the meaning. Hence we give a definition of resolved chaos for 1-parameter family of unimodal maps as follows.

Definition 7. "resolved chaos"

Let $\{f_\lambda\}_{\lambda=0}^{+\infty}$ be a 1-parameter family of unimodal maps. We say this family is with resolved chaos if it satisfies

(i) $\lim_{\lambda\to+\infty} N(f_\lambda) = +\infty$ and

(ii) There exists a positive δ_0 such that for all $\delta\in[0,\delta_0)$

$\overline{\lim}_{\lambda\to+\infty} \Delta n(f_\lambda,\delta) \leqq 1.$

The degree of chaos of $\{f_\lambda\}$ increases as $\lambda\to+\infty$ by the assumption (i) and sojourning time is predictable for a fluctuation smaller than δ_0 as $\lambda\to+\infty$ by the assumption (ii).

Proposition 4-1

Let $\{f_\lambda\}$ be a family with resolved chaos, then under the notation of Definition 7, we have $\underline{\lim}_{\lambda\to+\infty} T(f_\lambda) \geqq \delta_0$.

We will study some sufficient conditions for resolved chaos. Hereafter we assume that a unimodal map f satisfies $k=1$, $f(1)=1$ and $f(x)>0$ for all $x\in\mathbb{R}_+$ through this section. f is necessarilly a unimpdal map of type(A) in Definition 1 from this assumption.

We consider a 1-parameter family $\{f_\lambda\}$ defined by $f_\lambda(x)=\lambda\cdot f(x)$. Next theorem connects the decreasing order of f with the increasing order of $N(f_\lambda)$.

Theorem 4-2.

Let $a_1=\sup_{x\in(0,1]} f(x)/x$ and $a_2=\inf_{x\in(0,1]} f(x)/x$. If $a_1<+\infty$ and $a_2>0$, then we have

$$1+\underline{\lim}_{\lambda\to+\infty}\frac{\log\lambda}{\log f(\lambda)} \leqq \underline{\lim}_{\lambda\to+\infty}\frac{N(f_\lambda)}{-\log f(\lambda)/\log\lambda} \quad \text{and}$$

$$\overline{\lim}_{\lambda\to+\infty}\frac{N(f_\lambda)}{-\log f(\lambda)/\log\lambda} \leqq 1.$$

Especially, $\lim_{\lambda\to+\infty} N(f_\lambda) = +\infty$ if and only if $\lim_{\lambda\to+\infty} -\dfrac{\log f(\lambda)}{\log\lambda} = +\infty$ and if $\lim_{\lambda\to+\infty} N(f_\lambda) = +\infty$, then $\lim_{\lambda\to+\infty}\dfrac{N(f_\lambda)}{-\log f(\lambda)/\log\lambda} = 1$.

We think a little wider class of 1-parameter families of unimodal maps. Let $f_{\lambda,k}(x) = \lambda\cdot k\cdot f(x/k)$, then the turning point of $f_{\lambda,k}$ is k and $N(f_{\lambda,k}) = N(f_\lambda)$. We get a 1-parameter family $\{F_\lambda\}$ by letting k be dependent on λ and defining $F_\lambda=f_{\lambda,k(\lambda)}$.

Now we study some conditions on f and $k(\lambda)$ which ensure that $\{F_\lambda\}$ is with resolved chaos.

Theorem 4-3.

We assume f is twice differentiable at 0.

Let $\delta_1=\underline{\lim}_{\lambda\to+\infty}\lambda\cdot k(\lambda)$. If one of two conditions

(I) $\underline{\lim}_{\lambda\to+\infty} k(\lambda) > 0$ and $\lim_{\lambda\to+\infty} -\dfrac{\log f(\lambda)}{\lambda(\log\lambda)^2} = 0$

(II) $k(\lambda)\downarrow 0$ as $\lambda\uparrow+\infty$ and $\lim_{\lambda\to+\infty} -\dfrac{\log f(\lambda)}{\lambda\cdot k(\lambda)(\log\lambda)^2} = 0$

is satisfied, then we have $\overline{\lim}_{\lambda\to+\infty}\Delta n(F_\lambda,\delta) \leqq 1$ for all $\delta\in[0,\delta_1)$.

One can easily verify next corollary of Theorem 4-3 by Theore 4-2.

Corollary 4-4.

If $N(F_\lambda)\to+\infty$ as $\lambda\to+\infty$, then the conditions (I) and (II) are equivalent to (I)' and (II)' respectively.

(I)' $\quad \underline{\lim}_{\lambda\to+\infty} k(\lambda) = 0$ and $\lim_{\lambda\to+\infty} \dfrac{N(F_\lambda)}{\lambda\cdot\log\lambda} = 0.$

(II)' $\quad k(\lambda)\downarrow 0$ as $\lambda\uparrow+\infty$ and $\lim_{\lambda\to+\infty} \dfrac{N(F_\lambda)}{\lambda\cdot k(\lambda)\log\lambda} = 0.$

Remark.

We consider the case where both λ and k depend on a parameter t. Let $\lambda(t)\to+\infty$ as $t\to+\infty$, then Theorem 4-3 is also true i we replace the limit for $\lambda\to+\infty$ by the limit for $t\to+\infty$.

It is obious from Corollary 4-4 that if $\lim_{\lambda\to+\infty} N(F_\lambda)=+\infty$, (I)' and (II)' leads f to be with resolved chaos. Recalling Theorem 3 we may consider that (I)' and (II)' indicate that resolved manner (a 1-parameter family depends on the increasing order of entropy t the parameter of the family.

§5 An example from population biology.

We consider a population model by R. May[3] such that

$$X_{n+1} = X_n \exp(r(1-X_n/c))$$

where X_n is a population of n-th year, r is a growth rate and c is a carrying capacity. Fixing the carrying capacity and letti the growth rate to infinity, we have a 1-parameter family of unimodal maps $\{F_r\}$ where $F_r(x) = x\cdot\exp(r(1-x/c))$. Instead of thi: family we can consider a family $\{f_\lambda\}$ defined by

$$f_\lambda(x) = \lambda \cdot x \cdot \exp(1 - x/k(\lambda)), \quad k(\lambda) = \frac{c}{1+\log \lambda}.$$

(If $c=1$, then $f_1(x)=x\cdot\exp(1-x)$ satisfies the assumptions in §4. For example, 1 is the turning point of f_1 , $f_1(1)=1$, $a_1<+\infty$, $a_2>0$, f_1 is twice differentiable at 0.) We examine the results in §4 to this family.

(i) $\lim\limits_{\lambda\to+\infty} -\log f_1(\lambda)/\log \lambda = +\infty$

Hence we see $\lim\limits_{\lambda\to+\infty} N(f_\lambda) = +\infty$ from Theorem 4-2.

(ii) $\delta_1 = \varliminf\limits_{\lambda\to+\infty} \lambda\cdot k(\lambda) = +\infty$ and

$\lim\limits_{\lambda\to+\infty} -\log f_1(\lambda)/\lambda\cdot k(\lambda)(\log \lambda)^2 = 0$

Hence we see $\varlimsup\limits_{\lambda\to+\infty} \Delta n(f_\lambda,\delta) \leqq 1$ for all $\delta \geqq 0$ from Theorem 4-3.

Therefore $\{f_\lambda\}$ (i.e. $\{F_r\}$) is with resolved chaos. We also get $\lim\limits_{r\to+\infty} T(F_r) = +\infty$ from Proposition 4-1.

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