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Application of Fourier Hyperfunction to Fourier Expansion in Continuous Crossed product

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Abstract. This is an attempt to the application of the formulation of Fourier hyperfunction to a problem of operator algebra. Let $(M,\mathbb{R},\alpha)$ be a separable continuous W*-dynamical system. Then any element in the continuous crossed product $M\times_\alpha\mathbb{R}$ can be expressed as a "operator valued Fourier hyperfunction" on $\mathbb{R}$.

§1. Introduction

Among the various operations in operator algebras, crossed product is one of the most important tool not only for the construction of examples but also for the structure analysis of type III W*-algebras (see [1],[2],[8],[9]). Let (M,G,α) be a W*-dynamical system. If G is discrete, then the generic element of $M\times_\alpha G$ is expressible as a M-valued sequence indexed by G i.e. M-valued function on G. But in general, it is so difficult to give a desireble expression of the element of $M\times_\alpha G$ is G is not discrete.

In this note, we have an intension to offer one trial to answer the following problem of Fourier expansion.

Problem. Are there any suitable way to give an expression of the generic element of the continuous crossed product $M\times_{\alpha}G$ of W*-dynamical system (M,G,α) as a "function" on G with values in some topological vector space similar to M ?

In the preceding discussions, we only consider the case $G = \mathbb{R}$ and further assume that the predual M_* is separable. Throughout this note, we use the standard notion of continuous crossed product (for example, see [9]) and the discussion on standard form (see [3]). We also use the formulation of Fourier hyperfunction and its vector valued version, see [4], [5],[6].

Remark. The problem of Fourier expansion for $G = \mathbb{R}$ is already discussed by H. Takai [7] based on the analysis of the predual of the crossed product $M\times_{\alpha}\mathbb{R}$ and distribution theory. Our approach is different from this direction.

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§2. Twisted Plancherel Transformation

In this section, we consider W*-dynamical system (M,G,α) with locally compact abelian group G and present some discussions on Plancherel transformation for later use. The crossed product $M\times_\alpha G$ associated with the W*-dynamical system (M,G,α) is a W*-algebra generated by the following two families of operators on a Hilbert space $L^2(G)\otimes H$:

$$(2.1)\quad [\pi(x)\xi](g) = \alpha_{-g}(x)\xi(g),\ x\in M,$$

$$(2.2)\quad [\lambda(h)\xi](g) = \xi(-h+g),\ g,h\in G,\ \xi\in L^2(G)\otimes H,$$

where H is some representation Hilbert space of M. Let $\mathfrak{A} = K(G,M)$ be the set of all M-valued continuous (with respect to the *-strong topology on M) functions on G with compact support. Then $\mathfrak{A}$ is a *-algebra by

$$(2.3)\quad [f_1*f_2](g) = \int_G f_1(h)\alpha_n[f_2(-h+g)]dh,$$

$$(2.4)\quad f^*(g) = \alpha_g[f(-g)^*],$$

where the measure on G is the Plancherel measure associated with G. There exists a *-homomorphism $\tilde{\pi}\colon \mathfrak{A} \to B(L^2(G)\otimes H)$ given by

$$(2.5)\quad [\tilde{\pi}(f)\xi](g) = \int_G \alpha_{-g}[f(h)]\xi(-h+g)dh.$$

The crossed product $M\times_\alpha G$ is also given by the weak closure of $\tilde{\pi}(\mathfrak{A})$ (for example, see [9]). Now, we take H to be a standard representation Hilbert space of M so that the action α is implemented by a unitary representation U of G on H i.e. $\alpha_g(x) = U(g)xU(g)^*$, $g\in G$, $x\in M$ (see [3]). We define a twisted Plancherel transform $\xi\in L^2(G)\otimes H \to \hat{\xi}\in L^2(\hat{G})\otimes H$ by

$$(2.6)\quad \hat{\xi}(k) = \int_G \langle\overline{k,g}\rangle U(g)\xi(g)dg,\ k\in\hat{G}.$$

Let $\mathcal{B} = K(G,B(H))$ be the set of all $B(H)$-valued continuous functions on G with compact support, where $B(H)$ is the set of all bounded linear operators on the Hilbert space H. We define a mapping $f\mapsto \hat{f}$, $f\in\mathcal{B}$ by

$$(2.7)\quad \hat{f}(k) = \int_G \langle\overline{k,g}\rangle f(g)U(g)dg,\ k\in\hat{G},$$

where the integral in (2.7) is in the sense of weak integral. Then by Riemann-Lebesgue theorem, $\hat{f}\in L^\infty(\hat{G})\otimes B(H)$ and we obtain the following facts.

Lemma 2.1.

(1) $(f_1 * f_2)\hat{}\,(k) = \hat{f}_1(k)\cdot\hat{f}_2(k)$, $f_1,f_2\in\mathfrak{A}$.

(2) $\hat{f}^*(k) = \hat{f}(k)^*$, $f\in\mathfrak{A}$.

(3) $(\tilde{\pi}(f)\xi,\zeta) = (\pi_d(\hat{f})\hat{\xi},\hat{\zeta})$, $f\in\mathfrak{A},\xi,\zeta\in L^2(G)\otimes H$,

where $[\pi_d(\hat{f})\hat{\xi}](k) = \hat{f}(k)\hat{\xi}(k)$.

(4) $\|\tilde{\pi}(f)\| = \operatorname{ess.sup}_{k\in G}\|\hat{f}(k)\|$.

(5) $\int_{\hat{G}} \hat{f}(k)\hat{\xi}(k)dk = \int_G f(g)\xi(-g)dg$, $f\in\mathfrak{A}$, $\xi\in L^2(G)\otimes H$.

<u>Proof</u>. (1)~(3) and (5) follow from the definition. (4) follows from (3) and the unitarity of the twisted Plancherel transformation $\xi \mapsto \hat{\xi}$. Q.E.D.

§3. Embedding into $F_\tau(\mathbb{R},M)$

Now we consider the special case $G = \mathbb{R}$. Here, we apply the formulation of Fourier hyperfunction. Let T be the self-adjoint operator on H satisfying $U(t) = \exp(itT)$, $t \in \mathbb{R}$, and $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ be the spectral family of T i.e.

$$(3.1)\quad T = \int_{\mathbb{R}} \lambda dE(\lambda).$$

According to this decomposition, we set

$$(3.2)\quad H_\lambda = E[-\lambda,\lambda]H,\ \lambda > 0.$$

For a triple of positive real numbers $\Lambda = (\varepsilon,\lambda,\mu)$, we denote by $\mathcal{O}_\Lambda$ the set of all H_λ-valued function ϕ on $U_\varepsilon \equiv \mathbb{R}\times\sqrt{-1}[-\varepsilon,\varepsilon]$ which is

(a) holomorphic in the interior of U_ε,

(b) continuous on U_ε,

(c) $\sup_{z\in U_\varepsilon} \|\phi(z)\|e^{\mu|z|} \equiv \|\phi\|_\Lambda < \infty$.

Then we obtain the following facts.

<u>Lemma 3.1</u>.

(1) $\mathcal{O}_\Lambda$ <u>is a Banach space by the norm</u> $\|\cdot\|_\Lambda$.

(2) <u>There exists an injection</u> $\mathcal{O}_\Lambda \to L^2(\mathbb{R})\otimes H$ <u>with a</u>

<u>dense image</u>.

(3) <u>The (usual) Fourier transformation</u> $\phi \mapsto \tilde{\phi}$ <u>defined by</u>

$$(3.3) \quad \tilde{\phi}(k) = \int_{\mathbb{R}} e^{-ikt}\phi(t)dt$$

<u>induces a mapping</u> $\mathcal{O}(\varepsilon,\lambda,\mu) \to \mathcal{O}(\mu,\lambda,\varepsilon)$.

(4) <u>The mapping</u> $\phi \mapsto U\phi$ <u>defined by</u> $[U\phi](t) = U(t)\phi(t)$ <u>induces a mapping</u> $\mathcal{O}(\varepsilon,\lambda,\mu) \to \mathcal{O}(\varepsilon,\lambda,\mu-\lambda)$.

(5) <u>Let</u> $\varepsilon_1 \geqq \varepsilon_2$, $\lambda_1 \geqq \lambda_2$, $\mu_1 \geqq \mu_2$. <u>Then the natural mapping</u> $\mathcal{O}(\varepsilon_1,\lambda_1,\mu_1) \to \mathcal{O}(\varepsilon_2,\lambda_2,\mu_2)$ <u>is a norm decreasing injection</u>.

<u>Proof</u>. (1),(2),(5) follow from the definition. (3) follows from the property of Fourier transformation, (4) follows from the definition of H_λ. Q.E.D.

If we define the order $(\varepsilon_1,\lambda_1,\mu_1) \leqq (\varepsilon_2,\lambda_2,\mu_2)$ by $\varepsilon_1 \leqq \varepsilon_2$, $\lambda_1 \geqq \lambda_2$, $\mu_1 \leqq \mu_2$, then by Lemma 3.1 (5), the family $\{\mathcal{O}_\Lambda\}$ is a projective system of Banach spaces (all mappings are norm decreasing embedding). Now, we define a locally convex topology on $\mathcal{B} \supset \mathfrak{M}$, for which we denote by τ, determined by the following family of seminorms:

$$(3.4) \quad \Phi^{\tau}_{\phi,\xi}(f) = \left|\left(\int_{\mathbb{R}} f(t)\phi(t)dt,\xi\right)\right|, \ \phi \in \mathcal{O}_\Lambda, \ \xi \in H.$$

<u>Definition 3.2</u>. We define $F_\tau(\mathbb{R},B(H))$ (resp. $F_\tau(\mathbb{R},M)$) to be the sequential closure of $\mathcal{B}$ (resp. $\mathfrak{M}$) with respect to τ-topology.

Lemma 3.3. The twisted Plancherel transformation (for $G = \mathbb{R}$) extends to give an automorphism of $F_\tau(\mathbb{R}, B(H))$.

Proof. By Lemma 3.1 (3),(4), the family $\{\mathcal{O}_\Lambda\}$ is invariant under the twisted Plancherel transformation (2.6). By Lemma 2.1 (5), $\Phi^\tau_{\hat{\phi},\hat{\xi}}(\hat{f}) = \Phi^\tau_{\tilde{\phi},\xi}(f)$, where $\tilde{\phi}(t) = \phi(-t)$. Hence we obtain the assertion. Q.E.D.

Proposition 3.4. There exists a continuous injective linear mapping $M \times_\alpha \mathbb{R} \to F_\tau(\mathbb{R}, M) \subset F_\tau(\mathbb{R}, B(H))$.

Proof. By the separability of M_*, H is separable and hence, $L^2(\mathbb{R}) \otimes H$ is separable. It follows that the $*$-strong density of $\tilde{\pi}(\mathfrak{A})$ in $M \times_\alpha \mathbb{R}$ implies the weakly sequentially density of $\tilde{\pi}(\mathfrak{A})$ in $M \times_\alpha \mathbb{R}$. Asuume that $\{f_n\}$ is a sequence in $\mathfrak{A}$ such that zero Cauchy in τ-topology and Cauchy in $\tilde{\pi}$-topology, where $\tilde{\pi}$-topology is defined by the family of seminorms $f \mapsto |(\tilde{\pi}(f)\xi_1, \xi_2)|$, $\xi_1, \xi_2 \in L^2(\mathbb{R}) \otimes H$. Then $\{\hat{f}_n\}$ is a zero Cauchy sequence in τ-topology by Lemma 3.3 and a Cauchy sequence in π_d-topology by Lemma 2.1 (3), where π_d-topology is defined by the family of seminorms $\hat{f} \mapsto |(\tilde{\pi}_d(\hat{f})\hat{\xi}_1, \hat{\xi}_2)|$, $\hat{\xi}_1, \hat{\xi}_2 \in L^2(\mathbb{R}) \otimes H$. By Banach-Steinhaus theorem and by the fact that $\{\hat{f}_n\}$ is a Cauchy sequence in π_d-topology, $\{\hat{f}_n\}$ is a uniformly bounded sequence in $L^\infty(\mathbb{R}) \otimes B(H)$. By the definition of τ-topology, the family of seminorms defined by (3.4) separates $L^\infty(\mathbb{R}) \otimes B(H)$, we conclude that $\{\hat{f}_n\}$ is a zero Cauchy sequence and hence $\{f_n\}$ is a zero Cauchy sequence by Lemma 3.3. Hence the identity mapping of $\mathfrak{A}$ extends to give an injection $M \times_\alpha \mathbb{R} \to F_\tau(\mathbb{R},M)$. Q.E.D.

§4. Embedding into tensor product space

In this section, we show the existence of continuous injection $F_\tau(\mathbb{R},M) \to R(\mathbb{D})\hat{\otimes}_\varepsilon \tilde{M}$, where $R(\mathbb{D})$ is the space of Fourier hyperfunction on $\mathbb{R}$, and $\tilde{M}$ is a topological vector space obtained by the sequential closure of M with respect to the family of seminorms $x \mapsto P_{\xi,\eta}(x) = |(x\xi,\eta)|$, $\xi \in H_\lambda$, $\lambda \in \mathbb{R}$, $\eta \in H$. It is easily seen that $C_c(\mathbb{R})\otimes_{alg}M$ is sequentially dense in $\mathfrak{A}$ with respect to π-topology, where $C_c(\mathbb{R})$ is the set of all continuous function on $\mathbb{R}$ with compact support. Hence, $C_c(\mathbb{R})\otimes_{alg}M$ is sequentially dense in $F_\tau(\mathbb{R},M)$ with respect to τ-topology.

<u>Proposition 4.1</u>. <u>There exists a continuous injective linear mapping</u> $F_\tau(\mathbb{R},M) \to R(\mathbb{D})\hat{\otimes}_\varepsilon \tilde{M}$.

<u>Proof</u>. We define a locally convex topology on $C_c(\mathbb{R})\otimes_{alg}$ M, for which we denote by τ^w, determined by the following family of seminorms:

(4.1) $\Phi^w_{\phi,\xi,\eta}(f) = |\langle f, \phi\otimes\omega_{\xi,\eta}\rangle|$, $\phi \in \mathcal{O}(\varepsilon,\mu)$, $\xi \in H_\lambda$, $\eta \in H$,

where $\mathcal{O}(\varepsilon,\mu)$ is the function space defined in Section 3 with H_λ replaced by $\mathbb{C}$ and $\omega_{\xi,\eta}$ is a linear form on M defined by $\omega_{\xi,\eta}(x) = (x\xi,\eta)$, $x \in M$. Then by the similar argument as the proof of Proposition 3.4, we obtain a continuous injective linear mapping from $F_\tau(\mathbb{R},M)$ to the sequential closure of $C_c(\mathbb{R})\otimes_{alg}M$ with respect to τ^w-topology. By making use of the bipolar theorem, it is seen that the seminorms of τ^w-topology coincide with the seminorms which give rise to the ε-tensor product of $R(\mathbb{D})$ and $\tilde{M}$. Q.E.D.

Combining Propositions 3.4 and 4.1, we obtain a continuous injective linear mapping $M\times_\alpha\mathbb{R} \to R(\mathbb{D})\hat{\otimes}_\varepsilon\tilde{M}$ i.e. any element of $M\times_\alpha\mathbb{R}$ is expressible as a "$\tilde{M}$-valued Fourier hyperfunction" on $\mathbb{R}$.

§5. Discussions

The results obtained in this note is not so concrete, only presenting the trial to attack the problem of Fourier expansion in continuous crossed product. The remaining problem is the following.

(1) Which space is more convenient to handle, $F_\tau(\mathbb{R},M)$ or $R(\mathbb{D})\hat{\otimes}_\varepsilon\tilde{M}$? Are there any more "good" space in which we can embed $M\times_\alpha\mathbb{R}$?

(2) Characterize the image of the embedding map in $F_\tau(\mathbb{R},M)$ (resp. $R(\mathbb{D})\hat{\otimes}_\varepsilon\tilde{M}$) and write down the formula of *-algebraic operation.

(3) Is it possible to replace $\tilde{M}$ by a more concrete space?

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