数理解析研究所講究録
509巻 1984年 48-50

Aposyndesis and cut points which are related to refinable maps

Hiroshi Hosokawa (細川洋)
Department of Mathematics, Tokyo Gakugei Univ.

1. Results. We shall fix a refinable map $r : X \to Y$ between continua (compact connected metric spaces). Let y be a point of Y. There is a $(1/n)$-refinement r_n of r for each positive integer n such that $\{r_n^{-1}(y)\}$ converges to some point $\hat{y}$ of X (see [1]).

Theorem 1. If X is aposyndetic (resp. semi-aposyndetic, mutually aposyndetic, semi-locally connected, locally remotely connected) at $\hat{y}$, then so is Y at y.

A closed subset F of a space M is said to separate (resp. weakly separate) M if $M - F$ is not connected (not continuumwise connected). When F consists of only one point, $F = \{p\}$, then p is said to be a cut point (a weak cut point) of M provided that $\{p\}$ separates (weakly separates) M.

Theorem 2. (i) A point y of Y is a weak cut point of Y if and only if $r^{-1}(y)$ weakly separates X.

(ii) If Y is semi-locally connected at y, then y is a cut point of Y if and only if $r^{-1}(y)$ separates X.

2. Proof of Theorem 1. This Theorem is a pointwise version of the results in [2] except for the last case.

First let us assume that X is mutually aposyndetic at $\hat{y}$.

Let z be a point of $Y - \{y\}$. We may assume that $\{r_n^{-1}(z)\}$ converges to some point $\hat{z}$ of X. Since $\hat{y} \neq \hat{z}$, there are disjoint continuum neighborhoods H and K of $\hat{y}$ and $\hat{z}$ in X respectively. Choose an integer n so that $r_n^{-1}(y) \subset \operatorname{int}(H)$ and $r_n^{-1}(z) \subset \operatorname{int}(K)$. Then $r_n(H)$ and $r_n(K)$ are disjoint continuum neighborhoods of y and z in Y respectively. Therefore Y is mutually aposyndetic at y.

Secand let us assume that X is locally remotely connected at $\hat{y}$ (i.e. each neighborhood of $\hat{y}$ contains an open neighborhood of $\hat{y}$ whose complement is connected). Let U be a given neighborhood of y in Y and let V be an open neighborhood of y such that $\overline{V} \subset U$. Since $r^{-1}(V)$ is a neighborhood of $\hat{y}$, there is an open neighborhood V_0 of $\hat{y}$ in $r^{-1}(V)$ such that $X - V_0$ is connected. Choose an integer n so large that it satisfies $r_n^{-1}(y) \subset V_0$ and $1/n < d(Y - U, \overline{V})$. Then $U_0 = Y - r_n(X - V_0)$ is an open neighborhood of y in U such that $Y - U_0$ is connected.

The remaining cases can be proved by slight modifications of the proof of the case of mutual aposyndesis.

3. Proof of Theorem 2. (i). Let us assume that y is not a weak cut point of Y. Then there is a sequence $\{K_m\}$ of subcontinua of Y such that $K_1 \subset K_2 \subset \ldots$, and $Y - \{y\} = \bigcup_{m=1}^{\infty} K_m$. Inductively we can choose a sequence of subsequences $\{r_{m,n}\}_{n=1}^{\infty}$, $m = 1, 2, \ldots$, of $\{r_n\}_{n=1}^{\infty}$ such that

(1) $\{r_{m+1,n}\}_{n=1}^{\infty}$ is a subsequence of $\{r_{m,n}\}_{n=1}^{\infty}$,

(2) $\{r_{m,n}^{-1}(K_m)\}_{n=1}^{\infty}$ converges to some continuum $\hat{K}_m$ for $m = 1, 2, \ldots$.

The sequence $\{\hat{K}_m\}_{m=1}$ is increasing and satisfies $X - r^{-1}(y) = \bigcup_{m=1}^{\infty} K_m$. Hence $r^{-1}(y)$ does not weakly separates X.

(ii). Assume that y is not a cut point of Y. Since Y is semi-locally connected at y, Whyburn's Theorem implies that y is not a weak cut point. Hence Y is locally remotely connected at y. Suppose that $r^{-1}(y)$ separates a from b in X. There are open neighborhoods V_1 and V_2 of y such that $\overline{V}_2 \subset V_1$, $\overline{V}_1 \subset Y - \{r(a), r(b)\}$, and that $Y - V_1$ is connected. We may assume that $\{r_n^{-1}(\overline{V}_1)\}$ and $\{r_n^{-1}(\overline{V}_2)\}$ converge to some closed sets K_1 and K_2 respectively. Since $\mathrm{int}(K_2)$ separates a from b in X, there is a separation $X - \mathrm{int}(K_2) = A \cup B$, where $a \in A$ and $b \in B$. Choose an integer n so that $1/n < \min\{d(A, B), d(\overline{V}_2, Y - V_1), d(r(a), \overline{V}_1), d(r(b), \overline{V}_1)\}$. It is easy to see that $Y - V_1 = (r_n(A) - V_1) \cup (r_n(B) - V_1)$ is a separation of $Y - V_1$. This contradiction implies that $r^{-1}(y)$ does not separate X.

The "only if" parts are easy to prove.

References

[1] J. Ford and J. W. Rogers, Refinable maps, Colloq. Math.,39 (1978),263-269.

[2] H. Hosokawa, Aposyndesis and coherence of continua under refinable maps, to appear in Tsukuba J. Math.,5(1983).