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<th>The Fixed Point Property for Continua with Finitely Many Arc Components (STUDIES ON CONTINUOUS AND INFINITE-DIMENSIONAL MANIFOLDS)</th>
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<td>Author(s)</td>
<td>Tominaga, Akira</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1984), 509: 41-47</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1984-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/103786">http://hdl.handle.net/2433/103786</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>Publisher</td>
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The Fixed Point Property for Continua
with Finitely Many Arc Components

Akira TOMINAGA (富 永 晃)

1. Introduction. A space $X$ has the fixed point property (FPP) if each map $f:X \rightarrow X$ leaves some point fixed — that is, there is a point $x \in X$ such that $f(x) = x$. Let $X$ be a space approximated from within by subsets with FPP. Then it is natural to ask when $X$ has FPP.

In [5] Young showed that if $X$ is an arcwise connected Hausdorff space such that every monotone increasing sequence of arcs is contained in an arc, then $X$ has FPP. While Borsuk [1] proved that every dendroid has FPP. Afterward Ward [3] generalized both of these results as follows: Let $X$ be a chained acyclic Hausdorff space. Suppose that there is a point $e \in X$ such that for every ray $R$ with initial point $e$ the set $\bigcap_x (R - \{e, x\})$ has FPP. Then $X$ has FPP.

In Section 2 of this article we show a sufficient condition that a continuum, approximated from within by Peano continua with FPP, has FPP. As a consequence we have that for every positive integer $n$ the Cartesian product of $n$ Warsaw circles is a $T^n$-like continuum with FPP and the $n$-fold suspension of Warsaw circle is an $S^n$-like continuum with FPP, where $T^n$, $S^n$ mean an $n$-dimensional torus, an $n$-sphere, respectively. Here refer that Dyer [2] proved that the Cartesian product of $n$ chainable continua has FPP.

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In Section 3 we consider FPP for continua with two or more arc components.

2. Continua approximated from within by Peano continua with FPP.

**DEFINITION.** A continuum is a compact connected metric space and a Peano continuum is a locally connected continuum. A map is a continuous function. Let \((M, d)\) be a metric space and \(\varepsilon\) a positive number. A map \(f: M \rightarrow M\) is said to be \(\varepsilon\)-near to the identity map or simply to be \(\varepsilon\)-near if \(d(x, f(x)) < \varepsilon\) for every \(x \in M\).

**THEOREM 1.** Let \(X\) be a continuum for which there exists a sequence \(C_1 \subseteq C_2 \subseteq \cdots\) of Peano continua such that \(X = \bigcup_i C_i\) and every \(C_i\) has FPP. If the following (1) and (2) hold, then \(X\) has FPP.

(1) For every \(\varepsilon > 0\), there exist a \(C_i\) and a function \(f: X \rightarrow X\) such that for each \(s > i\) the restriction \(f|_{C_s}\) is an \(\varepsilon\)-near map of \(C_s\) to \(C_i\).

(2) There exists a closed subset \(A\), which may be empty, of \(\bigcap_j \overline{X - C_j}\), such that every \(f\) in (1) is continuous on \(A\), \(f(A) \subseteq C_1\) and every point \(x \in \bigcap_j \overline{X - C_j}\) - \(A\) has a neighborhood \(U\) whose component containing \(x\) lies in a \(C_i\).

**PROOF (SKETCH).** Let \(g: X \rightarrow X\) be an arbitrary map. For every \(C_i\) and every \(\delta > 0\), there exists a \(C_s\) with \(g(C_i) - N_\delta(A) \subseteq C_s\), where \(N_\delta(A)\) is a \(\delta\)-neighborhood of \(A\) in \(X\).
Then for every $C_i$, we have $f \circ g(C_i) \subseteq C_i$, and there exists an $x \in X$ with $g(x) = x$.

**Remark.** An $n$-sphere $S^n$ ($n \geq 1$) satisfies condition (1) in Theorem 1 but has no FPP.

**Theorem 2.** Suppose that for each $k$ ($1 \leq k \leq n$) $X_k$ and $C_{k1} \subseteq C_{k2} \subseteq \ldots$ satisfy the conditions in Theorem 1. If every $C_{1i} \times C_{2i} \times \ldots \times C_{ni}$ ($i = 1, 2, \ldots$) has FPP, then so does $X_1 \times X_2 \times \ldots \times X_n$.

To prove this it is sufficient to show that $X = X_1 \times \ldots \times X_n$ and $C_i = C_{1i} \times \ldots \times C_{ni}$ ($i = 1, 2, \ldots$) satisfy the conditions in Theorem 1.

**Definition.** Let $X$ and $Y$ be compact metric spaces. Then $X$ is said to be $Y$-like if for every $\varepsilon > 0$ there is a map $f$ of $X$ onto $Y$ such that for every $y \in Y$ the diameter of $f^{-1}(y)$ is less than $\varepsilon$.

**Corollary 1.** The Cartesian products of $n$ Warsaw circles is a $T^n$-like continuum with FPP for $1 \leq n \leq \omega$.

**Corollary 2.** The Cartesian product of the above $T^n$-like continuum and an $m$-cell ($1 \leq m \leq \omega$) has FPP.

For a set $P$, the symbols $P^\#$ and $P^*$ denote the cone.
over $P$ and the suspension of $P$, respectively.

**THEOREM 3.** Assume that $X$ and $C_1 \subset C_2 \subset \ldots$ satisfy the conditions in Theorem 1. If every $C_i^\# (C_i^*) (i = 1, 2, \ldots)$ has FPP, then so does $X^\# (X^*)$.

To prove this, it is sufficient to show that $X^\# (X^*)$ and $C_1^\# \subset C_2^\# \subset \ldots (C_1^* \subset C_2^* \subset \ldots)$ satisfy the conditions of Theorem 1. In this case the vertex of $X^\#$ and the suspension points of $X^*$ correspond to the set $A$ in Theorem 1.

**COROLLARY 3.** For every positive integer $n$, the $n$-fold suspension of Warsaw circle is an $S^n$-like continuum with FPP. Also the Cartesian product of this continuum and an $m$-cell ($1 \leq m \leq \omega$) has FPP.

**REMARK.** Recently Watanabe [4] obtained fixed point theorems for cones over certain general spaces.

3. Continua with finitely many arc components.

**DEFINITION.** A finite set $T$ is called an ordered tree if (1) $T$ is the set of vertices of a one-dimensional polyhedron containing no simple closed curve, and (2) $T$ is a partially ordered set such that a pair $p, q$ of points is the vertices of an edge if and only if one of them covers the other. (By "$p$ covers $q$" in a partially ordered set $\{P, \_\_\}$, it is meant that $p > q$, but that $p > x > q$ for no $x \in P$.) A function $f: P \rightarrow Q$ between partially ordered sets is isotone if $f(x) \geq$
f(y) whenever x \geq y. A partially ordered set P has the
fixed point property (FPP) if every isotone function f:P \rightarrow P
leaves an element of P fixed, i.e., there exists an x \in P
with f(x) = x.

LEMMA 1. Let P be a partially ordered finite set. If
there exists a maximum or minimum element in P, then P has
FPP.

The case where P has a minimum element follows from
Knaster-Tarski's theorem.

LEMMA 2. Every ordered tree has FPP.

This follows from induction on the number of elements of T.
Let g be a collection of mutually exclusive subsets G_\lambda
of a
topological space X such that \bigcup_\lambda G_\lambda = X. Then we define a
binary relation \leq on g as follows: For G_\lambda, G_\mu \in g, G_\lambda \leq
G_\mu if and only if there exists a finite sequence G_1, G_2, ..., G_k
of elements of g such that G_1 = G_\lambda, G_k = G_\mu and \bar{G}_i \cap G_{i+1}
\neq \emptyset (1 \leq i \leq k). The relation \leq is not necessarily a partial
order.

LEMMA 3. Let G_1, G_2 be arc components of a space X,
and let f:X \rightarrow X be a continuous map. If f(G_1) \cap G_2 \neq \emptyset,
then f(G_1) \subset G_2.

Let g be the collection of arc components of a topological
space $X$. If $f: X \to X$ is continuous, then for every $G_\lambda \in g$
there exists a $G_\mu \in g$ with $f(G_\lambda) \subset G_\mu$. Thus we define a
function $f^*: g \to g$ by $f^*(G_\lambda) = G_\mu$. From Lemma 3 we have

**Lemma 4.** If $G_1 \leq G_\kappa$, then $f^*(G_1) \leq f^*(G_\kappa)$.

**Theorem 4.** Let $X$ be a continuum with finitely many arc
components $G_1, G_2, \ldots, G_n$ such that each $G_i$ has FPP. If $g =
\{G_1, G_2, \ldots, G_n; \leq\}$ is an ordered tree or a partially ordered
set with a maximum or minimum element, then $X$ has FPP.

This follows from Lemmas 4, 2 and 1.

**Corollary 4.** Let $X$ be the continuum in Theorem 4. Let $Y$
be an arcwise connected continuum such that each $\overline{G_i} \times Y$ has
FPP. Then $X \times Y$ has also FPP.

**Theorem 5.** Let $X$ be a continuum with finitely many arc
components $G_1, G_2, \ldots, G_n$ satisfying the following conditions:

1. For every $i$ there exists a monotone increasing se-
quence $C_{i1} \subset C_{i2} \subset \cdots$ of subsets of $G_i$ such that $G_i = \bigcup_j C_{ij}$
has FPP.

2. $\bigcap_j G_i - C_{ij} = \overline{G_i} - G_i$ ($1 \leq i \leq n$).

3. $g = \{G_1, G_2, \ldots, G_n; \leq\}$ is an ordered tree each of
whose elements is covered by at most one element.

Then $X$ has FPP.

**Proof (Sketch).** Let $f: X \to X$ be a map. Then by Lemma 2
there exists an $a$ with $f(G_a) \subseteq G_a$. If $G_a$ is the maximum element of $g$, then $G_a = C_{s_j}$ for some $j$, and hence $f$ leaves a point of $C_{s_j}$ fixed. Suppose that $G_s$ is not the maximum element. If $f(C_{s_j}) \subseteq C_{s_j}$ for some $j$, then there exists a fixed point of $f$ in $C_{s_j}$. If for every $j$, $f(C_{s_j})$ is not contained in $C_{s_j}$, then there exist a point $x_0 \in X$ and $G_t$ such that $x_0 \cup f(x_0) \subseteq \overline{G_s} - G_s \subseteq G_t$. Therefore we have $f(G_t) \subseteq G_t$. Continuing this process, we can find a fixed point of $f$.

**COROLLARY 5.** Let $X$ be the continuum in Theorem 5. Let $Y$ be an arwise connected continuum such that $C_{i_j} \times Y$ (1 $\leq i \leq n$, $j = 1, 2, \ldots$) have FPP. Then $X \times Y$ has FPP.

**REFERENCES**


Faculty of Integrated Arts and Sciences, Hiroshima University, Hiroshima, JAPAN

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